FLUID FLOW AND HEAT TRANSFER IN THE END CAP OF A COAXIAL, RETURN-FLOW HEAT EXCHANGER UNDER STOKES CONDITIONS

BY

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by

Lotfi Ammar

To my parents:

Zohra & Salah

To my wife:

Yasmina

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LIST OF SYMBOLS

The dimensions of the symbols are given in terms of Mass (M), Length (L), Time (t), Temperature (T), Force ($F \equiv ML/t^2$) and Energy ($E \equiv ML^2/t^2$).

$A_{2n-1}, B_{2n-1}, \ldots, S_{2n-1}$	Coefficients of differential equations in 3.36
$a^{(2)}_{2n-1}, \ldots, c^{(0)}_{2n+1}$	Coefficients in Equation 3.28, are functions of x, and n
a	Equatorial radius of the spheroid, (L)
c	Polar radius of the spheroid, (L)
d	Feeder tube radius, (L)
g_{2n-1}	Dimensionless functions of x
G_{2n-1}	Dimensionless functions of x, defined in 3.25
h	Step size in the x direction, (L)
h	Heat transfer coefficient, (E/tT)
h_1, h_2, h_3	Scale factors in the oblate spheroidal coordinate system, dimensionless
k	Thermal conductivity, (E/tLT)
N	Number of differential equations in 3.36
Nu	Nusselt number, $Nu = \frac{h}{k}$, dimensionless
P_m	Legendre polynomial of degree m
p	Fluid pressure, (M/Lt^2)
p_m, q_m	Harmonic functions appropriate to the oblate spheroid

Pr	Prandtl number, $Pr=\nu/\alpha$, dimensionless
r,z	Orthogonal cylindrical coordinates, (L)
Re	Reynolds number, $Re=2\overline{V}d/\nu$, dimensionless
R	Radial distance in polar coordinates, (L)
R_{ij}	Coefficients of matrix $[\underline{R}]$ defined in D.12
S_{ij}	Coefficients of matrix $[\underline{S}]$ defined in D.14
T_{ij}	Coefficients of matrix $[\underline{T}]$ defined in D.10
T	Fluid temperature, (T)
T_w	Wall temperature of the feeder tube, (T)
T_s	Surface temperature of the cap, (T)
u	Velocity component in x direction, (L)
v	Velocity component in μ direction, (L)
$\overline{V}$	Mean fluid velocity in the feeder tube, (L/t)
x	Modified spheroidal coordinate component, dimensionless
x_j	Position at index j in x direction, dimensionless

GREEK SYMBOLS

α	Thermal diffusivity, (L^2/t)
α	Angle of polar coordinate system, Figure 6
β_{2n-1}	Coefficients, dimensionless
$\underline{\Gamma}_{2n-1}$	Column matrix of $g_{2n-1,j}$ coefficients
$\hat{\delta}_x, \hat{\delta}_\mu, \hat{\delta}_\phi$	Unit vectors in (x, μ, ϕ) system
ζ, μ	Oblate spheroidal coordinates, dimensionless
ζ_o	Dimension of the spheroidal surface, $\zeta = \zeta_o$
θ	Dimensionless fluid temperature
$\overline{\theta}$	Modified dimensionless temperature by Legendre transform
θ_o	Zeroth order solution of temperature field, dimensionless
θ_1	First order solution of temperature field, dimensionless
ν	Kinematic viscosity, (L^2/t)
ξ_1, ξ_2, ξ_3	Curvilinear coordinate system, dimensionless
ρ	Fluid density, (M/L^3)
ψ	Stream function, (L^3/t)
Ψ	Modified stream function, dimensionless
ω	Fluid vorticity $(1/t)$

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An analytical solution is developed for studying fluid flow and heat transfer in the end cap of a single-pass, return flow heat exchanger, under stokes conditions where the inertial effects are negligible in comparison with those of viscosity. The cap has the configuration of an oblate spheroid and the flow is considered to be in a steady state. A low speed flow of an incompressible medium is considered for which the stream function is expressed in a series of Legendre polynomials which converge rapidly for the ranges of the parameters studied in this work.

In the study of the heat transfer in the end cap, the temperature field is solved using a parameter perturbation method. The zeroth order perturbation corresponds to a pure conduction problem, and the convective effect is accounted for in the first order perturbation solution which is also the highest order considered in this work.

The computed stream functions are used to plot streamlines, which are, in turn, utilized to determine the velocity and pressure fields. The temperature fields are used to plot isotherms from which the local Nusselt number is also determined. The analysis reveals that there is a point of singularity for the vorticity at the rim of the feeder tube. A recirculation flow is also found when the annulus region has an area that is large compared to the feeder tube.

CHAPTER I

INTRODUCTION

Of the numerous heat exchangers developed in the past, the coaxial, return-flow heat exchanger is one of the most interesting. It consists of two coaxial tubes, one inside the other, see Figure 1. The round cap at one end serves as a guide for directing the flow from one passage to the other. There are many variations in which this exchanger can be used. The inner tube can be used as the feeder for fluid flow. Then, the annulus serves as the return channel. The fluid flow direction may be reversed using the annulus as the feeder and the inner pipe as the return passage. In addition, several flow patterns may develop within the passages depending on the tube dimensions. For example, if the radius of the outer pipe is large compared with that of the inner pipe, a recirculation flow appears near the inner end of the inner pipe. This circulation is not present for moderate radius ratios. Furthermore, if the inner pipe extends toward the end cap such that a small gap is formed between the tube and the round cap, this obstructs the flow such that another flow condition prevails.

Notice that those conditions used to characterize the fluid flow and heat transfer inside tubes and annuli may also be used for the study of the coaxial exchanger. Thus, from the fluid flow point of view, there may be laminar or turbulent

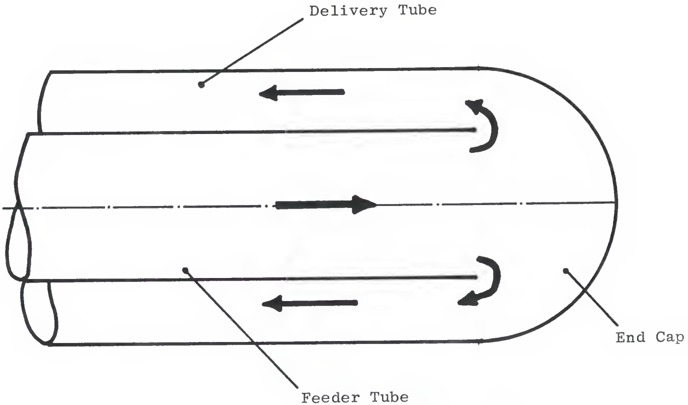


Figure 1. Coaxial, Return-Flow Heat Exchanger

flow, and developing and fully developed flow. On the other hand, from the heat transfer standpoint, there may be a prescribed temperature or a heat flux imposed as boundary conditions. The thermal boundary layer may be fully developed or may be in the developing stage. The coaxial, return-flow heat exchanger thus encompasses an array of problems for study.

It may be noted that the coaxial heat exchanger has been used by Owens-Illinois to develop the SUNPAK solar collector, see Figure 2. Coaxial heat exchangers are inserted into a common manifold at the center of the tube bank such that the fluid flow in the exchanger is connected in series. SUNPAK collectors use an evacuated envelope for the exchanger to reduce the heat loss. Their thermal efficiency is about 60% [1].

The coaxial heat exchanger has also been used in Alkylation contactors in which the exchangers form a tube bundle erected in a vertical position, see Figure 3. One of the advantages of this coaxial exchanger in this erected installation is that, since it is mounted at one end, it is free to expand at the other end, thus minimizing stress and seal problems.

While the exchanger has been considered useful for many applications, there has been little work done to study its fluid flow and heat transfer. This motivated the research given in this work. This dissertation presents an analytic solution. Part of the motivation for this approach is that numerical solutions (finite difference or finite element), while they can always be used for solution, must be tested for convergence. Such a test may be costly for a complicated

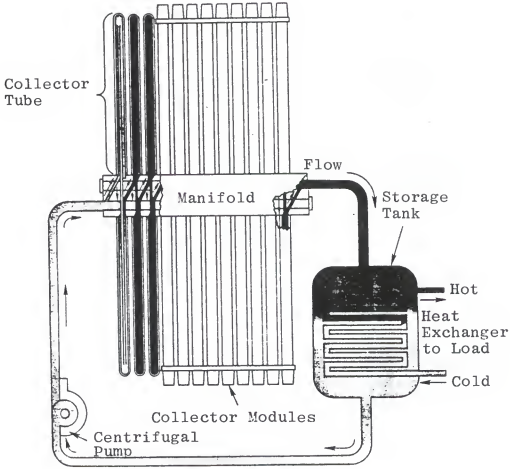


Figure 2. Sunpak Series Collectors

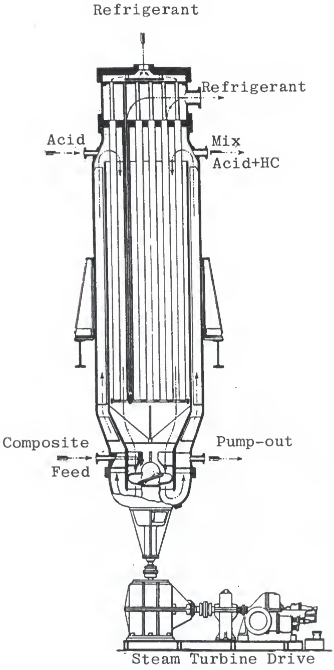


Figure 3. Field Tubes in an Alkylation Contactor

system like the heat exchanger. It is necessary to accept certain restrictions in order to achieve an exact solution. However, it does not suffer from the convergence. Notice also that the fluid flow and heat transfer inside tubes and annuli have been thoroughly studied and well documented in the literature, only the end cap poses as a new problem for analysis. The exact solution is thus dedicated to that, and this also defines the scope of this work.

In the chapters that follow, a literature survey will be provided in Chapter II. The formulation of the problem and the procedure for solution that leads to the determination of the stream function, and consequently the velocity and the pressure distributions in the cap, is presented in Chapter III. In Chapter IV an analysis of the heat transfer problem is provided for the solution of low Prandtl number situations. The results and discussions are summarized in Chapter V. Finally, some conclusions and recommendations are made in Chapter VI.

CHAPTER II

LITERATURE REVIEW

The coaxial, return-flow heat exchanger may be structurally divided into three components: the inner feed pipe, the annulus, and the end cap. Studies of fluid flow and heat transfer in these three components will be briefly reviewed in this chapter.

It is noted that the inner pipe and its surrounding annulus are two components in which the fluid flow and heat transfer problems have been thoroughly studied in the literature. Review monographs are also available for these systems. Hence, their review given here will be brief, and their inclusion is mainly for the sake of completeness. For an orderly presentation of the materials that follow, those monographs will be cited first. This will be followed by a brief review of the important papers published dealing with these subjects.

Fluid Flow and Heat Transfer in Tubes

Fluid flow and heat transfer in tubes and ducts are a topic of typical textbooks, e.g. [2-5]. Review of studies of fluid flow and heat transfer in ducts may be found in a series of monographs [6]. The most comprehensive review was published by Shah and London [5] in 1978.

A review of the literature reveals that the first analytical solution for heat transfer in a closed conduit was prescribed by Graetz [7] in 1883. Nusselt made an independent analytic solution in 1910 [8]. Dryden et al. [9] compiled solutions for fully developed laminar flow inside ducts of various geometries, in 1932.

Three methods of analysis have been applied to the flow in the entrance region. One method applies the integral form of the equations of momentum and mass continuity to the boundary layer as it develops along the tube wall. The velocity profile is approximated by a suitable polynomial as in the Karman-Pohlhausen method.

The second method subdivides the entrance region into two zones. In the zone near the tube entrance, a boundary-layer equation is used. An approximate solution is obtained by means of a perturbation series which takes the fully developed velocity distribution as a limiting case in the second region where the flow moves far downstream from the entrance section. The flow development throughout the entrance region is found by patching together the upstream and downstream solutions at some intermediate location.

In the third method, a solution for the velocity is obtained that is continuous radially and axially from the entrance to the fully developed region. Langhaar [10] in 1942 linearized the Navier-Stokes equations and solved to obtain a family of velocity profiles in the form of Bessel functions. The parameter of this family was tabulated against the axial coordinate in dimensionless form, and as a result, the transition length was also obtained. Sparrow et al. [11] linearized the momentum equation using the mean velocity along the flow direction. Sparrow's solution appears to be more general since it is applicable to both a circular pipe and a parallel-plate channel. A further improvement was made by Fleming and Sparrow [12] in 1969 allowing application to the entrance region of ducts of arbitrary cross section.

Analysis for heat transfer in a circular tube with fully developed velocity and temperature profiles is straight forward when the wall temperature or the heat flux is uniform. For an arbitrary circumferential heat flux but constant axial heat input, a superposition method can be used for solution [13]. When the flow is fully developed but heat transfer is in the developing stage, the problem can be solved by separation of variables. A specific example of this type is the case of uniform surface temperature when axial heat condution can be neglected; the solution in this case is in the form of Graetz functions. The eigenvalues for the infinite series in this solution can be obtained by numerical approximation. These methods have also been attempted by Sellars et al. [14] in the study of similar problems with uniform heat flux as boundary conditions.

The thermal entry length solutions described above form building blocks for studying heat transfer in pipes with surface temperature variations. Duhumel's theorem is useful in this instance. A similar analysis can be used to study problems with axial heat flux variations. However, the problem becomes more complicated when both velocity and temperature profiles develop simultaneously. The Prandtl number plays a leading role in this problem, and numerical methods are often used for solution.

Fluid Flow and Heat Transfer in Annuli

Murakawa [15] solved the continuity and momentum equations for the velocity distribution, pressure drop, and entry length for developing flow inside a circular annulus. An approximate solution was also developed by him for studying heat transfer in the thermal entry length. Hatton and Quarmby [16] developed a solution to the Graetz problem in an annulus with a heated core and an insulated outer wall. Both uniform temperature and uniform heat input on the inner wall were considered for a wide range of arbitrarily chosen radius ratios. Their solution was also extended to the case of axial variation of the core heat input and to the case in which heat was transferred at uniform but unequal rates at both walls.

The most comprehensive study of heat transfer in an annulus was made by the group led by Kays and Reynolds at Standford. A series of four papers was published by them, with the first one including a bibliography covering those annuli papaers published before 1961. Four classes of problems were formulated in this first paper [17], which consisted of (i) a step temperature on one wall, with the other wall maintained at the inlet temperature, (ii) a step heat flux from one wall, with the other wall kept at an adiabatic condition, (iii) a step temperature on one wall, with the other wall insulated, and finally (iv) a step heat flux from one wall, with the other maintained at the inlet temperature. The utility of these four basic problems was also described in this paper along with a superposition technique useful for the solution of more complicated problems. This paper was concluded with a brief description of the experiments which they used to verify analysis.

In the second paper [18], solutions for those four basic problems were developed for the case of a fully developed flow. In the analysis, the viscous generation of heat and the axial heat conduction effect were neglected in the energy equation. The temperature distributions, obtained from the solution of eigenvalue problems, were verified with experiments. Superposition of solutions of basic problems was also illustrated by means of examples.

In the third paper [19], heat transfer with turbulent flow was considered. The flow was taken to be fully developed inside an axisymmetrically heated annulus. Fluid properties were treated as constant, and axial heat conduction was again neglected in the analysis. Asymptotic solutions (fully developed velocity and temperature profiles) were developed for a wide range of radius ratios, Reynolds numbers, and Prandtl numbers. The analysis was made on the basis of empirical velocity and eddy diffusivity profiles, and experiments were carried out to verify the analysis at Pr=0.7. A superposition technique was also demonstrated for solving problems with axisymmetric heating from both walls of the annulus.

Heat transfer for laminar flow in an annulus with simultaneously developing velocity and temperature profiles was studied by Heaton et al. [20] in the final paper in the series. A solution was obtained first for the hydrodynamic problem, and then for the combined hydrodynamic and thermal problem with an integral method. Excellent agreement was again found between the experiment and the analysis.

As a further extension to what was published earlier in the series, Sutherland and Kays [21] studies heat transfer in an annulus with variable circumferential heat flux but constant heat rate along the tube axis. The velocity and temperature profiles were fully developed, and the analysis was carried out for both laminar and turbulent flows.

It is noted that, for those studies cited above, the fluid properties have been treated as constant. Shumway and McEligot [22] studied heat transfer in an annulus with temperature dependnent transport properties. Numerical solutions were carried out for both fully developed and uniform entry velocity profiles.

Fluid Flow and Heat Transfer in End Caps

Unlike the pipe and the annulus, there has been little attention given to the study of fluid flow and heat transfer in the end cap. There is a lack of literature published dealing directly with the fluid flow problem. However, a related problem of magnetohydrodynamic flow cap has been investigated [23,24]. In this magnetohydrodynamic problem, the fluid is set in motion due to the Lorentz force resulting from the electric current and the associated magnetic field.

The case of flow field induced in an incompressible viscous fluid was studied for discharging either from a point source [25] or a plane electrode [26] located at the center of the plane side of the hemisphere. For the point source problem, an analytic solution has been developed for the slow viscous flow and a numerical solution for the nonlinear problem. As one might expect, because of the axisymmetry of the position of the electrode, stream lines in an axial cross section form two closed loops. For the plane electrode case, the vorticity has a singularity at the rim of the electrode. When the width of the annulus, that is the remaining of the plane side of the hemisphere, is small relative to that of the electrode, an eddy is formed about the rim of the electrode. This eddy diminishes in size as the annulus width is increased.

Fluid Flow and Heat Transfer in the Total System

There have been few studies published for fluid flow and heat transfer in the total system of the coaxial exchanger. Hurd [27] studied the mean temperature difference for the exchanger used in the Alkylation contactor in which the inner feed pipe, or the annulus, was used to deliver the coolant and was thus at the shell side of the contactor. Based on the relative directions of flows between the annulus fluid and its shell-side, heat exchanging fluid, four flow conditions may arise. Studies were made by Hurd to develop the mean temperature difference equations and curves for thse four conditionss. Hurd's paper centers on the study of the application aspect of the heat exchanger. It provides little insight into the mechanisms of fluid flow and heat transfer.

More recently Moan [28] and Beekley and Mather [1] published performance data of coaxial exchangers fitted with evacuated envelopes to reduce the heat loss. The exchanger was investigated for solar energy applications, and the analysis presented was focused on the total system behavior.

CHAPTER III

ANALYSIS OF THE FLUID FLOW PROBLEM

Governing Equations

Consider a steady, incompressible flow of a viscous fluid ejecting from a circular tube of radius d into an oblate hemispheroidal cap and delivered through an annular passage of outer radius a, see Figure 4. The equitorial and polar radii of the spheroid are a and c, respectively. The axes of tubes lie on the axis of the cap, which is also the axis of symmetry of the heat exchanger.

Let V be the fluid velocity. Under the condition of constant properties, no body force, and sufficiently slow motion of the fluid such that the squares and products of the velocity components are negligible, the continuity equation and the Navier-Stokes equations can be simplified to be

$$\nabla \cdot V = 0 \tag{3.1}$$

$$-\nabla p + \rho\nu\nabla^2 V = 0 \tag{3.2}$$

where p, ρ, and ν have their usual meaning of pressure, density, and kinematic viscosity, respectively.

It is convenient to make use of the oblate spheroidal coordinates (ζ, u, ϕ) which are related to the cylindrical coordinates (r, z, ϕ) by

$$r = d(1 + \zeta^2)^{\frac{1}{2}}(1-\mu^2)^{\frac{1}{2}} \tag{3.3}$$

$$z = d\zeta\mu \tag{3.4}$$

Here the origin of the coordinate system (r, z, ϕ) is taken to be the center of the spheroid, and the positive z-axis is pointed toward the cavity side of the cap and is oriented along the axis of symmetry. Equations 3.3 and 3.4 are derived in Appendix A.

The inlet region can now be expressed as

$$r \leq d \text{ and } z = 0$$

in the cylindrical system, or

$$0 \leq \mu \leq 1 \text{ and } \zeta = 0$$

in the spheroidal system. Similarly, the annular passage is in the region

$$d \leq r \leq a \text{ and } z = 0$$

or

$$\mu = 0 \text{ and } 0 \leq \zeta \leq \zeta_o$$

The surface of the cap is given by $\zeta = \zeta_o$, $0 \leq \mu \leq 1$, and the fluid region by $0 \leq \mu \leq 1$, $0 \leq \zeta \leq \zeta_o$, see Figure 4. With the geometry given, the equatorial and polar radii of the spheroid can be related to d and ζ_o by

$$a = d(1+ \zeta_o^2)^{\frac{1}{2}} \text{ and } c = d\zeta_o$$

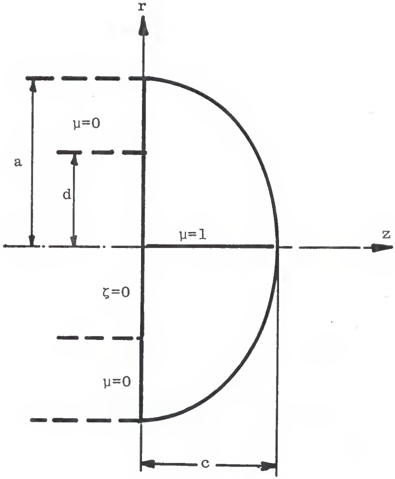


Figure 4. Section View of the Oblate Spheroid

Since the velocity field V is axisymmetric and thus two dimensional, it can be expressed in terms of the stream function $\Psi(\zeta,\mu)$ which satisfies the continuity equation identically. The velocity components u and v can be derived (see Appendix A) as

$$\vec{V} = \frac{1}{d^2(\zeta^2+\mu^2)^{\frac{1}{2}}} \left[\frac{-1}{(1+\zeta^2)^{\frac{1}{2}}} \frac{\partial\Psi}{\partial\mu} \, ; \, \frac{1}{(1-\mu^2)^{\frac{1}{2}}} \frac{\partial\Psi}{\partial\zeta} \right] \qquad (3.5)$$

Taking curl of Equation 3.2, and making use of vector identities $\nabla^2 V=\nabla(\nabla\cdot V)-\nabla x(\nabla x V)$ and $\nabla x \nabla p=0$ and also the continuity equation 3.1, it can be derived that

$$\nabla x \nabla x(\nabla x V) = 0$$

or (3.6)

$$\nabla x \nabla x \omega = 0$$

where ω is vorticity, defined as

$$\omega = \nabla x V \qquad (3.7)$$

Finally introducing $\Psi=\nu d\psi$ and $\zeta=\zeta_o x$ and making use of Equation 3.5, one can transform Equation 3.6 to

$$D^2 \frac{1}{(x^2+\mu^2\zeta_o^{-2})} D^2\Psi = 0 \qquad (3.8)$$

where

$$D^2 = (x^2+\zeta_o^{-2}) \frac{\partial^2}{\partial x^2} + (1-\mu^2) \frac{\partial^2}{\partial \mu^2}$$

Equation 3.8 becomes the transformed governing equation for the fluid flow problem.

Boundary Conditions

The governing Equation 3.8 will be solved subjected to a set of boundary conditions described below.

Along the axis of symmetry (i.e. $\mu=1$), the stream function must have a constant value. It is convenient to take this value to be zero. Physically, this characterizes the condition that the velocity component v, cross-flow is zero along this axis. At the spheroidal surface ($x=1$), refer to Figure 5, the velocity components u and v both vanish. Besides, this surface can be considered as an extension of the axis of symmetry ($\mu=1$) from the stream function standpoint. Hence, Ψ is zero on this surface.

For the low Reynolds number flow that is of interest in this work, the fluid velocity emerging from the feed pipe and exiting to the annulus region is expected to be nearly parallel to the axis of symmetry. It is common practice [29,30] to consider the velocity component along both $x=0$ and $\mu=0$ planes to be negligibly small, refer to Figure 5. Finally, the fluid velocity emerging from the feed pipe is assumed to be known [31,32]. The stream function can thus be determined at $x=0$.

These conditions can be summarized as follows

$$\Psi(x,\mu=1) = 0 \qquad \text{(axis of symmetry)} \qquad (3.9)$$

$$\Psi(x=1,\mu) = 0 \qquad \text{(spheroidal surface)} \qquad (3.10)$$

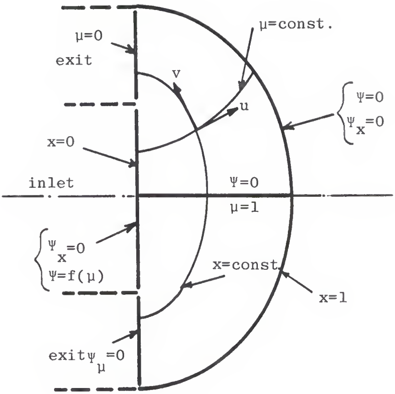


Figure 5. Boundary Conditions for the Fluid Flow Problem

$$\frac{\partial\Psi(x=1,\mu)}{\partial x} = 0 \qquad (3.11)$$

$$\frac{\partial\Psi(x=0,\mu)}{\partial x} = 0 \qquad \text{(inlet)} \qquad (3.12)$$

$$\Psi(x=0,\mu) = f(\mu) \qquad (3.13)$$

$$\frac{\partial\Psi(x,\mu=0)}{\partial\mu} = 0 \qquad \text{(exit)} \qquad (3.14)$$

They can also be put into perspective in Figure 5.

Development of a Solution Technique

The problem at hand can be solved by expanding the stream function in a series of Legendre polynomials

$$\Psi(x,\mu) = (1-\mu^2) \sum_{n=1}^{\infty} g_{2n-1}(x)\, P'_{2n-1}(\mu) \qquad (3.15)$$

Each term in this series satisfies Equations 3.9 and 3.14 identically. To satisfy Equations 3.10 through 3.12 calls for meeting new conditions as follows

$$g_{2n-1}(1) = 0 \qquad (3.16)$$

$$g'_{2n-1}(1) = 0 \qquad (3.17)$$

$$g'_{2n-1}(0) = 0 \qquad (3.18)$$

As mentioned earlier, Equation 3.13 signifies that the velocity is known at the exit of the feed pipe. It is possible to expand $f(\mu)$ in a series of $(1-\mu^2)P'_{2n-1}(\mu)$ over the region

$0 \leq \mu \leq 1$ provided that the function satisfies Dirichlet conditions in the interval $-1 \leq \mu \leq 1$ [33]. For the coaxial heat exchanger problem under investigation, let the velocity emerging from the feed pipe be fully developed. Then, the function $f(\mu)$ can be related to μ (see Appendix B) as

$$\Psi(0,\mu) = f(\mu) = \tfrac{1}{4}Re(1-\mu^4) \tag{3.19}$$

where the Reynolds number is defined as $2\overline{V}d/\nu$, and $\overline{V}$ refers to the mean velocity at the pipe exit.

In Equation 3.19, $f(\mu)$ can be expanded in a series following the pattern given in Equation 3.15. Thus,

$$\Psi(0,\mu) = (\tfrac{1}{2}Re)(1-\mu^2)\ [(3/5\ P_1'(\mu) + 1/15\ P_3'(\mu)] \tag{3.20}$$

A comparison of this equation with Equation 3.15 at x=0 provides the following conditions

$$g_1(0) = 3/10\ Re \tag{3.21}$$

$$g_3(0) = 1/30\ Re \tag{3.22}$$

and

$$g_{2n-1}(0) = 0 \qquad \text{for } n \geq 3 \tag{3.23}$$

These will be used as boundary conditions together with Equation 3.16 through Equation 3.18 in the solution for g. Notice that the function itself will satisfy a set of ordinary differential equations whose derivation follows next.

The stream function Ψ should satisfy Equation 3.8. Hence, the series of the Legendre polynomials 3.15 is substituted into

Equation 3.8, and after some manipulation (see Appendix C), giving

$$\sum_{n=1}^{\infty} (x^2+\zeta_o^{-2})\{[x^2 G''_{2n-1}(x)-4xG'_{2n-1}(x)]\, P'_{2n-1}(\mu)$$

$$+\zeta_o^{-2} G''_{2n-1}(x)\mu^2 P'_{2n-1}(\mu)\} - \{4\zeta_o^{-2}\mu[(1-\mu^2)P'_{2n-1}(\mu)]'$$

$$+ 2[(n+1)(2n-3)x^2-2\zeta_o^{-2}]P'_{2n-1}(\mu)$$

$$+2[(n+1)(2n-3)\zeta_o^{-2}+6\zeta_o^{-2}]\mu^2 P'_{2n-1}(\mu)\}G_{2n-1}(x) = 0 \qquad (3.24)$$

where

$$G_{2n-1}(x) = (x^2+\zeta_o^{-2})g''_{2n-1}(x)-2n(2n-1)g_{2n-1}(x) \qquad (3.25)$$

In order to determine G from this equation, it is appropriate to expand terms involving $\mu^2 P'_{2n-1}(\mu)$ and $\mu[(1-\mu^2)P'_{2n-1}(\mu)]'$ in terms of Legendre polynomials. Use is thus made of two identities [33]:

$$\mu^2 P'_{2n-1}(\mu) = \frac{2n(2n-1)}{(4n-1)(4n+1)} P'_{2n+1}(\mu) + \frac{(2n-1)+(2n-2)(4n+1)}{(4n+1)(4n-3)} P'_{2n-1}(\mu)$$

$$+ \frac{2n(2n-1)}{(4n-1)(4n-3)} P'_{2n-3}(\mu) \qquad (3.26)$$

$$\mu[(1-\mu^2)P'_{2n-1}(\mu)]' =$$

$$\frac{2n(2n-1)}{4n-1}\left[\frac{2n-1}{4n-3} P'_{2n-3}(\mu) - \frac{4n-1}{(4n+1)(4n-3)} P'_{2n-1}(\mu) - \frac{2n}{4n+1} P'_{2n+1}(\mu)\right]$$

$$(3.27)$$

which are substituted into Equation 3.24. The series in the resulting equation is then expanded, and finally those terms containing derivatives of the Legendre polynomials of the same degree (i.e. terms of P_1', P_3', etc. . . .) are grouped together and set to zero. A general formula is derived,

$$a^{(2)}_{2n-3}G''_{2n-3} + a^{(0)}_{2n-3}G_{2n-3} + b^{(2)}_{2n-1}G''_{2n-1} + b^{(1)}_{2n-1}G'_{2n-1} + b^{(0)}_{2n-1}G_{2n-1} + c^{(2)}_{2n+1}G''_{2n+1} + c^{(0)}_{2n+1}G_{2n+1} = 0 \qquad (3.28)$$

In this equation, coefficients a, b and c have been identified with both superscripts and subscripts to show where they belong. The superscripts are specifically used to relate the order of the derivatives of the G function with which they are multiplied. These coefficients are functions of ζ_o, n and x,

$$a^{(2)}_{2n-3} = (x^2+\zeta_o^{-2})\,\frac{2(n-1)(2n-3)}{(4n-3)(4n-5)}\,\zeta_o^{-2} \qquad (3.29)$$

$$a^{(0)}_{2n-3} = \frac{4(n-1)(2n-3)(-2n^2+9n-10)}{(4n-3)(4n-5)}\,\zeta_o^{-2} \qquad (3.30)$$

$$b^{(2)}_{2n-1} = (x^2+\zeta_o^{-2})(x^2+\frac{(8n^2-4n-3)}{(4n+1)(4n-3)}\,\zeta_o^{-2}) \qquad (3.31)$$

$$b^{(1)}_{2n-1} = -4x(x^2+\zeta_o^{-2}) \qquad (3.32)$$

$$b^{(0)}_{2n-1} = \frac{8n(2n-1)-2(2n^2-n+3)(8n^2-4n-3)}{(4n+1)(4n-3)}\,\zeta_o^{-2}$$

$$-2[(n+1)(2n-3)x^2-2\zeta_o^{-2}] \qquad (3.33)$$

$$c^{(2)}_{2n+1} = (x^2+\zeta_o^{-2})\,\frac{2(2n+1)(n+1)}{(4n+3)(4n+1)}\,\zeta_o^{-2} \qquad (3.34)$$

$$c^{(0)}_{2n+1} = \frac{-4(n+1)(n+2)(2n+1)(2n+3)}{(4n+3)(4n+1)}\,\zeta_o^{-2} \qquad (3.35)$$

and $a^{(2)}_{-1}$ and $a^{(0)}_{-1}$ are taken to be zero.

Equation 3.28 forms an infinite set of second order, linear differential equations; but, it was g conditions that were derived earlier. Hence, Equation 3.28 is changed to a differential equation in g by using the definition of G, Equation 3.25. This leads to a fourth order, linear differential equation of g as

$$A_{2n-1}g^{iv}_{2n-1} + B_{2n-1}g'''_{2n-1} + C_{2n-1}g''_{2n-1} + D_{2n-1}g'_{2n-1}$$
$$+ E_{2n-1}g_{2n-1} = -R_{2n-1} - S_{2n-1} \quad (n=1,\ 2,\ .\ .\ .\ ,\ N) \qquad (3.36)$$

where

$$R_1 = 0 \qquad (3.37)$$

$$R_{2n-1} = H_{2n-1}g^{iv}_{2n-3} + I_{2n-1}g'''_{2n-3} + J_{2n-1}g''_{2n-3}$$
$$+ K_{2n-1}g_{2n-3} \quad (n=2,\ 3,\ .\ .\ .\ ,\ N) \qquad (3.38)$$

$$S_{2n-1} = L_{2n-1}g^{iv}_{2n+1} + M_{2n-1}g'''_{2n+1} + N_{2n-1}g''_{2n+1}$$
$$+ Q_{2n-1}g_{2n+1} \quad (n=1,\ 2,\ .\ .\ .\ ,\ N-1) \qquad (3.39)$$

$$S_{2N-1} = 0 \qquad (3.40)$$

In the above formulation, the series is assumed to terminate at N. Also notice that the new coefficients A, B, . . . are related to the a, b, c coefficients mentioned earlier (Equations 3.29 through 3.35). The detailed derivation for these equations is given in Appendix C.

Several features in Equations 3.36 through 3.40 are worthy of note. R_{2n-1} and S_{2n-1} appear on the right hand side of Equation 3.36. Since R_{2n-1} is related to g_{2n-3} through Equation 3.38 and S_{2n-1} to g_{2n+1} through Equation 3.39 and the fact that the subscript n here designates the nth term in the series expansion, these g terms for succeeding expansions in the series are actually related according to Equation 3.36. Second, the derivations of g can be easily changed to finite differences in the numerical solution.

Equations 3.36 through 3.40 can then be expressed in terms of g(x), and these g(x) terms are, in turn, related to their neighboring g through these equations. It is now clear that these finite difference equations must be solved for the g function for each n in the series expansion, while these g functions for nth expansion are again related to their lower and higher order expanded terms. In this regard, one must be aware of the fact that, since g_{2n-1} is related to g_{2n-3} and g_{2n+1}, a tri-diagonal type matrix will result. On the other hand, since the highest order differentials in Equations 3.36, 3.38 and 3.39 are fourth order, g for each n is governed by a penta-diagonal matrix. With this in mind, a scheme for solution can be developed as follows (the details are again relegated to an Appendix D).

As the first step, the derivatives of g are expressed in finite difference forms, which are, in turn, substituted into Equations 3.36, 3.38 and 3.39. Using h to represent the step size in the x direction and j subscript to designate position in the x direction change these equations to the following

$$\begin{aligned}
&(A_{2n-1,j}-B_{2n-1,j}\,\tfrac{h}{2})g_{2n-1,j-2} + (-4A_{2n-1,j}+B_{2n-1,j}h \\
&+ C_{2n-1,j}\,h^{2}-D_{2n-1,j}\,\tfrac{h^{3}}{2})g_{2n-1,j-1} + (6A_{2n-1,j}-2C_{2n-1,j}\,h^{2} \\
&+ E_{2n-1,j}\,h^{4})g_{2n-1,j} + (-4A_{2n-1,j}-B_{2n-1,j}\,h+C_{2n-1,j}\,h^{2} \\
&+ D_{2n-1,j}\,\tfrac{h^{3}}{2})g_{2n-1,j+1} + (A_{2n-1,j}+B_{2n-1,j}\,\tfrac{h}{2})g_{2n-1,j+2} \\
&= -R_{2n-1,j}\,h^{4} - S_{2n-1,j}\,h^{4} \qquad (3.41)
\end{aligned}$$

$$\begin{aligned}
&(H_{2n-1,j}-I_{2n-1,j}\,\tfrac{h}{2})g_{2n-3,j-2} + (-4H_{2n-1,j}\,+I_{2n-1,j}\,h \\
&+ J_{2n-1,j}\,h^{2})g_{2n-3,j-1} +(6H_{2n-1,j}-2J_{2n-1,j}\,h^{2} +K_{2n-1,j}\,h^{4})g_{2n-3,j} \\
&+ (-4H_{2n-1,j}=I_{2n-1,j}\,h +J_{2n-1,j}\,h^{2})g_{2n-3,j+1} \\
&+ (H_{2n-1,j}+I_{2n-1,j}\,\tfrac{h}{2})g_{2n-3,j+2} = R_{2n-1,j}\,h^{4} \qquad (3.42)
\end{aligned}$$

$$\begin{aligned}
&(L_{2n-1,j}-M_{2n-1,j}\,\tfrac{h}{2})g_{2n+1,j-2} +(-4L_{2n-1,j}+M_{2n-1,j}\,h \\
&+ N_{2n-1,j}\,h^{2})g_{2n+1,j-1} +(6L_{2n-1,j}-2N_{2n-1,j}\,h^{2} + Q_{2n-1,j}h^{4})g_{2n+1,j} \\
&+ (-4L_{2n-1,j}-M_{2n-1,j}\,h+N_{2n-1,j}\,h^{2})g_{2n+1,j+1} \\
&+ (L_{2n-1,j}+M_{2n-1,j}\,\tfrac{h}{2})g_{2n+1,j+2} = S_{2n-1,j}\,h^{4} \qquad (3.43)
\end{aligned}$$

These equations can be slightly simplified by using boundary conditions 3.17 and 3.18. Consequently, for Equation 3.41, a penta-diagonal matrix results,

$$[\underline{T}]\ \{\underline{g_{2n-1}}\} = \begin{Bmatrix} 0 \\ -R_{2n-1,2}\ h^4\ -S_{2n-1,2}\ h^4 \\ \vdots \\ -R_{2n-1,M-1}\ h^4 - S_{2n-1,M-1}\ h^4 \\ 0 \end{Bmatrix} \qquad (3.44)$$

where n=1, 2, . . . ,N, and $[\underline{T}]$ is used to represent the penta-diagonal matrix. $\{\underline{g_{2n-1}}\}$ denotes a vector whose components are $g_{2n-1,1}$, g_{2n-1} , . . . ,$g_{2n-1,M}$. The subscript M here refers to the index for the end point, x=1.

In a similar fashion, Equation 3.42 can be expressed as

$$[\underline{R}]\ \{\underline{g_{2n-3}}\} = \begin{Bmatrix} 0 \\ R_{2n-1,2}\ h^4 \\ \vdots \\ R_{2n-1,M-1}\ h^4 \\ 0 \end{Bmatrix} \qquad (3.45)$$

where n=2, 3, . . . , N, and for Equation 3.43,

$$[\underline{S}]\ \{\underline{g_{2n+1}}\} = \begin{Bmatrix} 0 \\ S_{2n-1,2}\ h^4 \\ \vdots \\ S_{2n-1,M-1}\ h^4 \\ 0 \end{Bmatrix} \qquad (3.46)$$

where n=1, 2, . . .,N-1.

These three matrix equations can be combined to give

$$\begin{bmatrix} [\underline{T}]_{n=1} & [\underline{S}]_{n=1} & 0 \cdots & & 0 \\ [\underline{R}]_{n=2} & [\underline{T}]_{n=2} & [\underline{S}]_{n=2} & 0 \cdots & 0 \\ 0 & [\underline{R}]_{n=3} & [\underline{T}]_{n=3} & [\underline{S}]_{n=3} & \cdots 0 \\ \vdots & \vdots & & & [\underline{S}]_{n=N-1} \\ 0 & 0 & 0 \cdots & [\underline{R}]_{n=N} & [\underline{T}]_{n=N} \end{bmatrix} \begin{Bmatrix} \underline{\Gamma_1} \\ \underline{\Gamma_3} \\ \underline{\Gamma_5} \\ \vdots \\ \underline{\Gamma_{2N-3}} \\ \underline{\Gamma_{2N-1}} \end{Bmatrix} = 0 \tag{3.47}$$

where

$$\underline{\Gamma_1} = \begin{Bmatrix} g_{1,1} \\ g_{1,2} \\ \vdots \\ g_{1,M} \end{Bmatrix} \tag{3.48}$$

and

$$\underline{\Gamma_3} = \begin{Bmatrix} g_{3,1} \\ g_{3,2} \\ \vdots \\ g_{3,M} \end{Bmatrix} \tag{3.49}$$

In those matrices given above, $[\underline{T}]$, $[\underline{R}]$, and $[\underline{S}]$ are MxM matrices, and they are used in Equation 3.47 to construct the coefficient matrix which is itself an NxN matrix.

It is noted that the remaining boundary conditions to be satisfied are Equations 3.16, 3.21, 3.22 and 3.23. These equations are rewritten by introducing subscripts for g and summarized together with Equations 3.17 and 3.18 in Table 1. A close examination of this table reveals something interesting.

As always in this work, n (in the first column) designates the nth term in the series expansion. In the table, boundary conditions are given at both x=0 and x=1, that is, index 1 and M, respectively. Four conditions are given for each n, which are sufficient to solve the fourth order ordinary differential equation for g (Equation 3.36). However, as mentioned before, because of the appearance of R and S terms on the right hand side of this equation, $g_{1,1}$ is also related to $g_{3,1}$. By the same token, $g_{3,1}$ is related to $g_{1,1}$ and $g_{5,1}$, and $g_{5,1}$ to $g_{3,1}$ and $g_{7,1}$. Now, since g and g' for $n \geq 3$ all vanish at x=0 and x=1, it is expected that $g_7(x)$ will be zero everywhere. Furthermore, since $g_{3,1}$ is one ninth of that $g_{1,1}$ and $g_{5,1}$ is zero, $g_3(x)$ is expected to be smaller than $g_1(x)$ and $g_5(x)$ will be smaller still. That is to say, the Legendre polynomial expansion, Equation 3.15, will converge very rapidly with only three terms, at most, significant in the series expansion. Convergence of the solution is thus assured.

Special Treatment Near the Rim of the Feeder Tube

For flow near the rim of the feeder tube, it is convenient to consider another coordinate system (R,α) shown in Figure 6. Writing equations for ζ and μ as functions of R and α and subsequently expanding the resulting equations using Taylor's expansions permit writing (see Appendix E)

Table 1. A summary of boundary conditions for g(x)

n	2n-1	conditions given at x=0	conditions given at x=1
1	1	$g_{1,1}=3/10\ Re$, $g'_{1,1}=0$	$g_{1,M}=0$, $g'_{1,M}=0$
2	3	$g_{3,1}=1/30\ Re$, $g'_{3,1}=0$	$g_{3,M}=0$, $g'_{3,M}=0$
3	5	$g_{5,1}=0$, $g'_{5,1}=0$	$g_{5,M}=0$, $g'_{5,M}=0$
4	7	$g_{7,1}=0$, $g'_{7,1}=0$	$g_{7,M}=0$, $g'_{7,M}=0$
⋮	⋮	⋮ ⋮	⋮ ⋮
N	2N-1	$g_{2N-1,1}=0$, $g'_{2N-1,1}=0$	$g_{2N-1,M}=0$, $g'_{2N-1,M}=0$

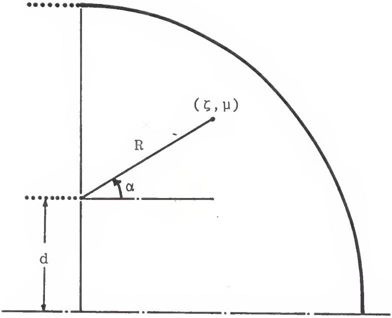


Figure 6. Coordinate System (R,α) at the Rim of the Feeder Tube

$$\zeta=(\frac{R}{d})^{\frac{1}{2}}[1+1/4(\frac{R}{d})+1/32(\frac{R}{d})^{2}(1-2\sin\alpha)+\ldots]\ (\sin\alpha/2+\cos\alpha/2) \quad (3.50)$$

$$\mu=(\frac{R}{d})^{\frac{1}{2}}[1-1/4(\frac{R}{d})+1/32(\frac{R}{d})^{2}(1+2\sin\alpha)+\ldots]\ (\cos\alpha/2-\sin\alpha/2) \quad (3.51)$$

for small R,

$$R \simeq \tfrac{1}{2}d(\zeta^{2}+\mu^{2}) \quad (3.52)$$

It is noted that the vorticity ω is related to ψ (see Equation A.7). Since $\Psi=\psi/\nu d$, and D^2 is defined as Equation A.11. ω can be expressed as

$$\omega = \frac{\nu d\ D^{2}\Psi}{d^{3}(1+\zeta^{2})^{\frac{1}{2}}(1-\mu^{2})^{\frac{1}{2}}(\zeta^{2}+\mu^{2})}\ \hat{\partial}_{\phi} \quad (3.53)$$

for small ζ and μ,

$$\omega \simeq \frac{\nu d\ D^{2}\Psi}{d^{3}(\zeta^{2}+\mu^{2})}\ \hat{\partial} \quad (3.54)$$

Upon using Equation 3.52,

$$\omega \simeq \frac{\nu d^{2}\ D^{2}\Psi}{d^{3}\ 2R} \quad (3.55)$$

This equation will be used later to show the singularity for ω at R=0.

It can be proved that, for small ζ (or x) and μ, the following expressions and their linear combinations are solutions to Equation 3.8:

$$1, \mu, x, \mu x, (\zeta_o^2 x^2 - \mu^2), \mu x^2, \mu^2 x, \mu^3, x^3, \mu^2 x^2, (\zeta_o^2 x^3 \mu - x\mu^3), \ldots$$

which are arranged in decreasing order of magnitude. If fourth order term is neglected in this array, it can be shown that one that is most dominant while satisfying Equations 3.12, 3.14, $\Psi(0,\mu)$=constant (coming from Equation 3.19 when μ is small), and $\Psi(x,0)\neq 0$ is x^3. Thus, near the rim,

$$\Psi = Ax^3 + B \tag{3.56}$$

where A and B are constants, which cannot be determined without reference to the overall solution for the stream function. Notice that this Ψ is compatible with the assumed series solution for Ψ (Equation 3.15). As will be shown later, $g_1(x)$ has a point of inflection, thus is of the type x^3. Equation 3.15 can be reduced to Equation 3.56 for n=1 under the condition of small μ.

Near the rim of the feeder tube, Ψ can be formulated by substituting Equation 3.50 into Equation 3.56. After some trigonometry,

$$\Psi = \frac{1}{2} \frac{A}{\zeta_o^3} \left(\frac{R}{d}\right)^{3/2} \left(1 + 3/4 \frac{R}{d}\right)(3\sin\alpha/2 + \sin 3\alpha/2 + 3\cos\alpha/2$$
$$-\cos 3\alpha/2) + B \tag{3.57}$$

This form of Ψ is actually similar to that for flow around the edge of a flat plate [34,35]. As is expected, for flow close to the rim of the feeder tube, the curvature of the tube becomes less important. Fluid flow around the feeder tube thus behaves much like that around a flat plate. It is also significant to note that if Equations 3.50 and 3.56 are substituted into Equation 3.55, ω can be related to R as $\omega \propto R^{-\frac{1}{2}}$, a point of singularity is found for ω at R=0.

Velocity and Pressure Fields

For fluid flow problems, the determination of the velocity and pressure fields is important. The stream function Ψ has been found and expressed in terms of the function g. Once this stream function is obtained along the stream line, the velocity field can be determined by using Equation 3.5. However, the evaluation of the pressure field is more complicated, as will be shown in this section.

The pressure field can be related to the vorticity by using Equations 3.1 and 3.2 and the vector identity $\nabla^2 V = \nabla(\nabla \cdot V) - \nabla x (\nabla x V)$. Since $\omega = \nabla x V$,

$$\nabla p = \rho \nu \nabla x \omega \tag{3.58}$$

∇p can be expressed in terms of scale factors which are defined in Appendix A,

$$\nabla p = \frac{1}{h_1} \frac{\partial p}{\partial \xi_1} \hat{\delta}_1 + \frac{1}{h_2} \frac{p}{\partial \xi_2} \hat{\delta}_2 \tag{3.59}$$

Using those relations between h, d and μ, and replacing ξ_1 and ξ_2 by $\xi_1 = d\zeta$ and $\xi_2 = \mu$ change Equation 3.59 to

$$\nabla p = \frac{1}{d(\zeta^2+\mu^2)^{\frac{1}{2}}} [(1+\zeta^2)^{\frac{1}{2}} \frac{\partial p}{\partial \zeta} \hat{\delta}_\zeta + (1-\mu^2)^{\frac{1}{2}} \frac{\partial p}{\partial \mu} \hat{\delta} \mu] \tag{3.60}$$

On the other hand, taking curl of ω given by Equation 3.53 yields

$$\nabla x\omega = \frac{1}{d^4(\zeta^2+\mu^2)^{\frac{1}{2}}}\left\{\frac{1}{(1+\zeta^2)^{\frac{1}{2}}}\frac{\partial}{\partial\mu}\left[\frac{D^2\psi}{(\zeta^2+\mu^2)}\right]\hat{\delta}_\zeta - \frac{1}{(1-\mu^2)^{\frac{1}{2}}}\frac{\partial}{\partial\zeta}\left[\frac{D^2\psi}{(\zeta^2+\mu^2)}\right]\hat{\delta}_\mu\right\} \tag{3.61}$$

The pressure gradient can then be derived by substituting Equations 3.60 and 3.61 into Equation 3.58,

$$\frac{\partial p}{\partial x} = \frac{-\rho\nu}{(d\zeta_o)^3(x^2+\zeta_o^{-2})}\frac{\partial}{\partial\mu}\left[\frac{D^2\psi}{(x^2+\mu^2\zeta_o^{-2})}\right] \tag{3.62}$$

$$\frac{\partial p}{\partial\mu} = \frac{\rho\nu}{(d\zeta_o)^3(1-\mu^2)}\frac{\partial}{\partial x}\left[\frac{D^2\psi}{(x^2+\mu^2\zeta_o^{-2})}\right] \tag{3.63}$$

Now, refer to Equation 3.15, $D^2\psi$ can be derived to be

$$D^2\psi = \nu d(1-\mu^2)\sum_{n=1}^{\infty} G_{2n-1}(x)P'_{2n-1}(\mu) \tag{3.64}$$

where $G_{2n-1}(x)$ has been given as Equation 3.25. Substituting Equation 3.64 into Equation 3.62 and Equation 3.63 gives

$$\frac{\partial p}{\partial x} = \frac{2\rho\,\nu^2 d}{(d\zeta_o)^3(x^2+\zeta_o^{-2})}\sum_{n=1}^{\infty} G_{2n-1}(x)\left[\frac{n(2n-1)}{(x^2+\mu^2\zeta_o^{-2})}P_{2n-1}(\mu) + \frac{\mu\,\zeta_o^{-2}(1-\mu^2)}{(x^2+\mu^2\zeta_o^{-2})}P'_{2n-1}(\mu)\right] \tag{3.65}$$

$$\frac{\partial p}{\partial \mu} = \frac{\rho\nu^2 d}{(d\zeta_o)^3} \sum_{n=1}^{\infty} P'_{2n-1}(\mu) \left[\frac{G'_{2n-1}(x)}{(x^2+\mu^2\zeta_o^{-2})} - \frac{2xG_{2n-1}(x)}{(x^2+\mu^2\zeta_o^{-2})^2} \right] \qquad (3.66)$$

These equations can be used together to find p. In this effort, say Equation 3.66 is used first to integrate with respect to μ to give

$$p(x,\mu) = \frac{\rho\nu^2}{(d\zeta_o)^3} \sum_{n=1}^{\infty} G_{2n-1}(x) \int_o^{\mu} \frac{P'_{2n-1}(\mu)}{(x^2+\mu^2\zeta_o^{-2})} d\mu$$

$$- 2xG_{2n-1}(x) \int_o^{\mu} \frac{P'_{2n-1}(\mu)}{(x^2+\mu^2\zeta_o^{-2})^2} d\mu + F(x) \qquad (3.67)$$

This equation is, in turn, differentiated with respect to x to give

$$\frac{\partial p}{\partial x} = \frac{\rho\nu^2 d}{(d\zeta_o)^3} \sum_{n=1}^{\infty} G''_{2n-1}(x) \int_o^{\mu} \frac{P'_{2n-1}(\mu)}{(x^2+\mu^2\zeta_o^{-2})} d\mu - 2 \Big[G_{2n-1}(x)$$

$$+2xG'_{2n-1}(x) \Big] \int_o^{\mu} \frac{P'_{2n-1}(\mu)}{(x^2+\mu^2\zeta_o^{-2})^2} d\mu + 8x^2 G_{2n-1}(x) \int_o^{\mu} \frac{P'_{2n-1}(\mu)}{(x^2+\mu^2\zeta_o^{-2})^3} d\mu$$

$$+ F'(x) \qquad (3.68)$$

Equation 3.68 can now be equated to Equation 3.65 to find $F'(x)$ at $\mu=0$ to be

$$F'(x)=0 \qquad (3.69)$$

which implies that

$$F(x)=\text{constant} \qquad (3.70)$$

Physically, this equation signifies that the pressure at the exit of the end cap (i.e., $\mu=0$) is a constant.

Equation 3.67 can be used to find the pressure distribution. Note that in this equation $G_{2n-1}(x)$ is related to $g_{2n-1}(x)$ by Equation 3.25, while $g_{2n-1}(x)$ is a function of ζ_o and the Reynolds number (through the boundary conditions of g). In view of the definition of the Reynolds number ($2\overline{V}d/\nu$), the geometry of the spheroidal cap has thus been totally accounted for.

A list of equations useful for the integrations in Equation 3.67 is compiled in Appendix F. An integrated version of this equation is also given there.

CHAPTER IV

ANALYSIS OF THE HEAT TRANSFER PROBLEM

Governing Equation

Consider a two dimensional system in steady state. The transport properties are treated as constant, and the dissipation effect is neglected in the analysis. Under these conditions, the energy equation is reduced to

$$(V\cdot\nabla)T = \alpha\nabla^2 T \tag{4.1}$$

where $\alpha = k/\rho c$ is the thermal diffusivity.

Making use of the oblate spheroidal coordinates (x,μ) and introducing the stream function through the use of velocity components given in Equation 3.5 transform Equation 4.1 to

$$\frac{\partial T}{\partial \mu}\frac{\partial \psi}{\partial x} - \frac{\partial T}{\partial x}\frac{\partial \psi}{\partial \mu} = \alpha d\zeta_o \left\{\frac{\partial}{\partial x}\left[(x^2+\zeta_o^{-2})\frac{\partial T}{\partial x}\right] + \frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial T}{\partial \mu}\right]\right\} \tag{4.2}$$

which has been derived in Appendix G. A change of ψ to Ψ recast Equation 4.2 as

$$\frac{\partial T}{\partial \mu}\frac{\partial \Psi}{\partial x} - \frac{\partial T}{\partial x}\frac{\partial \Psi}{\partial \mu} = \frac{\zeta_o}{Pr}\left\{\frac{\partial}{x}\left[(x^2+\zeta_o^{-2})\frac{\partial T}{\partial x}\right] + \frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial T}{\partial \mu}\right]\right\} \tag{4.3}$$

where $Pr=\nu/\alpha$ is the Prandtl number. Equation 4.3 is the energy equation to be solved for the fluid flow in the end cap.

Boundary Conditions

For flows of small Reynolds number with Prandtl numbers close to one, the conduction effect dominates. Under these conditions, the downstream condition usually has only slight effect on the temperature field in the upstream direction. A set of hypothetical conditions is thus chosen as follows

$$T(x=0,\mu)=T_w \qquad (4.4)$$

$$T(x=1,\mu)=T_s \qquad (4.5)$$

$$T(x,\mu=1)\neq\infty \qquad (4.6)$$

Development of a Solution Technique

In an effort to simplify the solution for the problem given, a dimensionless temperature is introduced as follows

$$\theta = \frac{T-T_s}{T_w-T_s}$$

Equation 4.3 is then changed to be

$$\frac{\partial\theta}{\partial\mu}\frac{\partial\Psi}{\partial x} - \frac{\partial\theta}{\partial x}\frac{\partial\Psi}{\partial\mu} = \frac{\zeta_o}{Pr}\left\{\frac{\partial}{\partial x}\left[(x^2+\zeta_0^{-2})\frac{\partial\theta}{\partial x}\right] + \frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\theta}{\partial\mu}\right]\right\} \qquad (4.7)$$

and for the boundary conditions,

$$\theta(x=0,\mu)=1;\ \theta(x=1,\mu) = 0 \qquad (4.8)$$

$$\theta(x,\ =1)\neq\infty;\ \theta(x,\mu=0) = \theta_o(x) \qquad (4.9)$$

where $\theta_o(x)$ is the solution of the one dimensional problem.

Notice that Equation 4.7 is linear and homogeneous in θ; so, the sum of solutions to Equation 4.7 is also an acceptable solution. This permits the use of a perturbation method in which the temperature θ is expanded in a parameter perturbation as

$$\theta = \sum_{j=o}^{\infty} (Pr)^j \ \theta_j(x,\mu) \tag{4.10}$$

The unperturbed part of the solution is that for j=0, the solution to a pure conduction problem. Equation 4.10 is a valid solution to Equation 4.7 provided that the parameter Pr is small.

Substituting Equation 4.10 into Equation 4.7 and equating like powers of Pr results in a differential equation that relates subsequent order solutions as

$$\nabla^2\theta_j = \frac{1}{\zeta_o}\left(\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial\mu} - \frac{\partial\Psi}{\partial\mu}\frac{\partial}{\partial x}\right)\theta_{j-1};\ j=1,2,3\ldots \tag{4.11}$$

where

$$\nabla^2 = \frac{\partial}{\partial x}\left[(x^2+\zeta_o^{-2})\frac{\partial}{\partial x}\right] + \frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial}{\partial\mu}\right]$$

The boundary conditions for subsequent θ_j can be derived in a similar manner as

$$\theta_o(x=0,\mu)=1,\ \theta_1(x=0,\mu) = \theta_2(x=0,\mu) = \ldots = 0 \tag{4.12}$$

$$\theta_o(x=1,\mu)= \theta_1(x=1,\mu) = \ldots = 0 \tag{4.13}$$

$$\theta_o(x,\mu=1)\neq\infty\ ,\ \theta_1(x,\mu=1)\neq\infty\ \ldots \tag{4.14}$$

Zeroth order solution

The zeroth order solution is derived for a pure conduction problem which satisfies

$$\frac{\partial}{\partial x}\left[(x^2+\zeta_o^{-2})\frac{\partial\theta_o}{\partial x}\right]+\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\theta_o}{\partial\mu}\right]=0 \qquad (4.15)$$

and

$$\theta_o(x=0,\mu)=1,\ \theta_o(x=1,\mu)=0,\ \theta_o(x,\mu=1)\neq\infty\ ,\ \frac{\partial\theta_o}{\partial\mu}(x,\mu=0)=0$$

Solution to Equation 4.15 has been given in [36] to be

$$\theta_o=\sum_{n=0}^{\infty}\left[A_n\,p_n(\zeta_o x)+B_n\,q_n(\zeta_o x)\right]P_n(\mu) \qquad (4.16)$$

where A_n and B_n are constants, and P_n are Legendre polynomials. p_n and q_n are

$$\begin{aligned} p_n(\zeta_o x) &= i^{-n}P_n(i\zeta_o x)\\ q_n(\zeta_o x) &= i^{-n}Q_n(i\zeta_o x)\end{aligned} \qquad (4.17)$$

Equation 4.13 can be used to establish

$$B_n=-A_n\,p_n(\zeta_o)/q_n(\zeta_o) \qquad (4.18)$$

It follows that

$$\theta_o=\sum_{n=o}^{\infty}A_n\left[P_n(\zeta_o x)-\frac{p_n(\zeta_o)}{q_n(\zeta_o)}\,q_n(\zeta_o x)\right]P_n(\mu) \qquad (4.19)$$

The second condition at x=0 (Equation 4.12) is used to derive A_n as

$$A_o = 1/[p_o(0) - \frac{p_o(\zeta_o)}{q_o(\zeta_o)} q_o(0)] \tag{4.20}$$

and $A_1 = A_2 = \ldots = 0$

In this manner the solution becomes

$$\theta_o = [p_o(\zeta_o x) - \frac{p_o(\zeta_o)}{q_o(\zeta_o)} q_o(\zeta_o x)] \;/\; [p_o(0) - \frac{p_o(\zeta_o)}{q_o(\zeta_o)} q_o(0)] \tag{4.21}$$

which can be rewritten as [36]

$$\theta_o = [1 - \frac{\cot^{-1}(\zeta_o x)}{\cot^{-1}\zeta_o}]/(1 - \frac{\pi}{2\cot^{-1}\zeta_o}) \tag{4.22}$$

First order solution

The first order solution must satisfy

$$\nabla^2\theta_1 = \frac{1}{\zeta_o}(\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial\mu} - \frac{\partial\Psi}{\partial\mu}\frac{\partial}{\partial x})\,\theta_o \tag{4.23}$$

with the boundary conditions

$\theta_1(x=0,\mu)=0,\ \theta_1(x=1,\mu)=0,\ \theta_1(x,\mu=1)\neq\infty$ and $\theta_1(x,\mu=0) = 0$

Clearly, the solution to Equation 4.23 now depends on Ψ and θ_o, which (Equations 3.14 and 4.22) are thus substituted into Equation 4.23 to give

$$\nabla^2 \theta_1 = \frac{4\zeta_o^{-2}}{(2\cot^{-1}\zeta_o - \pi)} \sum_{n=1}^{N} n(2n-1) \frac{g_{2n-1}(x)}{(x^2+\zeta_o^{-2})} P_{2n-1}(\mu) \quad (4.24)$$

The left hand side of Equation 4.24 can be expanded by using the general formula given in the appendix (Equation G.3). The resulting equation is multiplied by $P_\ell(\mu)$ and is integrated from μ=0 to 1 to give

$$\frac{d}{dx}\left[(x^2+\zeta_o^{-2}) \frac{d\overline{\theta}_\ell}{dx}\right] - P_\ell(0) \left[\frac{\partial \theta_1}{\partial \mu}\right]_{\mu=0} + P_\ell'(0)[\theta_1]_{\mu=0} - \ell(\ell+1)\overline{\theta}_\ell =$$

$$\frac{4\zeta_o^{-2}}{(2\cot^{-1}\zeta_o - \pi)} \sum_{n=1}^{N} n(2n-1) \frac{\zeta_{2n-1}(x)}{(x^2+\zeta_o^{-2})} \int_o^1 P_{2n-1}(\mu) P_\ell(\mu) d\mu \quad (4.25)$$

where

$$\overline{\theta}_\ell = \int_o^1 \theta P_\ell(\mu) d\mu \quad (4.26)$$

Equation 4.26 is known as the Legendre transform [37]. It can be noted that the boundary conditions chosen for θ_1 are all homogeneous and the nonhomogenity for θ is all left to the θ_o term, see Table 2. On the other hand, the gradient of θ_1 at μ=0 is not known; yet this lack of condition can be resolved by changing the subscript ℓ (for θ and P in Equation 4.25) to $2\ell-1$. Since $P_{2\ell-1}(0)$ is now zero, Equation 4.25 is reduced to

Table 2. A summary of boundary conditions for $\theta(x, \mu)$

zeroth order solution	first order solution
$\nabla^2\theta_o = 0$	$\nabla^2\theta_1 = \frac{1}{\zeta_o}\left(-\frac{\partial\Psi}{\partial\mu}\frac{\partial}{\partial x}\right)\theta_o(x)$
$\theta_o(x=0,\mu)=1$	$\theta_1(x=0,\mu)=0$
$\theta_o(x=1,\mu)=0$	$\theta_1(x=1,\mu)=0$
$\frac{\partial\theta_o}{\partial\mu}(x,\mu=0)=0$	$\theta_1(x,\mu=0)=0$
$\theta_o(x,\mu=1)\neq\infty$	$\theta_1(x,\mu=1)\neq\infty$

total solution

$$\theta(x,\mu)\equiv\theta_o(x,\mu)+\theta_1(x,\mu)$$

satisfies:

$$\nabla^2\theta=\nabla^2\theta_o+\nabla^2\theta_1 = \frac{1}{\zeta_o}\left(\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial\mu} - \frac{\partial\Psi}{\partial\mu}\frac{\partial}{\partial x}\right)\theta_o(x)$$

BC's:

$\theta(0,\mu)=\theta_o(0,\mu) + \theta_1(0,\mu) = 1 + 0 = 1$

$\theta(1,\mu)=\theta_o(1,\mu) + \theta_1(1,\mu) = 0 + 0 = 0$

$\theta(x,0)=\theta_o(x,0) + \theta_1(x,0) = \theta_o(x) + 0 = \theta_o(x)$

$\theta(x,1)=\theta_o(\neq\infty) + \theta_1(\neq\infty) \neq \infty$

$$\frac{d}{dx}\left[(x^2+\zeta_o^{-2})\,\frac{d\overline{\theta}_{2\ell-1}}{dx}\right] - 2\ell(2\ell-1)\overline{\theta}_{2\ell-1} = R_{2\ell-1}(x) \qquad (4.27)$$

where

$$R_{2\ell-1} = \frac{4\zeta_o^{-2}}{(2\cot^{-1}\zeta_o-\pi)} \sum_{n=1}^{N} n(2n-1)\,\frac{g_{2n-1}(x)}{(x^2+\zeta_o^{-2})}\int_o^1 P_{2n-1}(\mu)P_{2\ell-1}(\mu)d\mu \qquad (4.28)$$

Equation 4.28 can be further simplified by making use of the orthogonality properties of the Legendre polynomials; thus,

$$R_{2n-1} = \frac{4\zeta_o^{-2}}{(2\cot^{-1}\zeta_o-\pi)}\,\frac{n(2n-1)}{(4n-1)}\,\frac{g_{2n-1}(x)}{(x^2+\zeta_o^{-2})} \qquad (4.29)$$

Equation 4.27 can be solved by the method of variation of parameters. In this method, the solution to Equation 4.27 is assumed to be

$$\overline{\theta}_{2n-1}(x) = A_{2n-1}(x)\;p_{2n-1}(\zeta_o x) + B_{2n-1}(x)\;q_{2n-1}(\zeta_o x) \qquad (4.30)$$

where subscript ℓ has been changed to n. Equation 4.30 is differentiated to get

$$\frac{d\overline{\theta}_{2n-1}(x)}{dx} = A_{2n-1}(x)\;p'_{2n-1}(\zeta_o x) + B_{2n-1}(x)\;q'_{2n-1}(\zeta_o x) \qquad (4.31)$$

provided that

$$A'_{2n-1}(x)\;p_{2n-1}(\zeta_o x) + B'_{2n-1}(x)\;q_{2n-1}(\zeta_o x) = 0 \qquad (4.32)$$

In the above two equations, primes are used to designate differentiation of the primed quantity with respect to x.

It is possible to generate another relation between A and B by substituting Equation 4.31 into Equation 4.27, thus giving

$$P'_{2n-1}(\zeta_o x)A'_{2n-1}(x) + q'_{2n-1}(\zeta_o x)B'_{2n-1}(x) = \frac{R_{2n-1}(x)}{x^2+\zeta_o^{-2}} \tag{4.33}$$

Equations 4.32 and 4.33 can now be solved as a set of simultaneous equations for A' and B' to obtain

$$A'_{2n-1}(x) = \zeta_o^2 R_{2n-1}(x) q_{2n-1}(\zeta_o x) \tag{4.34}$$

$$B'_{2n-1}(x) = -\zeta_o^2 R_{2n-1}(x) p_{2n-1}(\zeta_o x) \tag{4.35}$$

These equations will again be solved simultaneously for A and B as follows.

Say, use is made of Equation 4.35 first, which is integrated from x=0 to x to get

$$B_{2n-1}(x) - B_{2n-1}(0) = -\zeta_o^{+2}\int_o^x R_{2n-1}(x) p_{2n-1}(\zeta_o x)dx \tag{4.36}$$

Equation 4.36 can be simplified by noting the fact that

$$\theta_1(x=0,\mu)=0$$

is the boundary condition for θ_1 at x=0. Applying Legendre transform of this condition provides

$$\overline{\theta}_{2n-1}(0)=0 \qquad (4.37)$$

Since $p_{2n-1}(0)=0$, Equation 4.30 can be used to get

$$B_{2n-1}(0)=0 \qquad (4.38)$$

Using this condition for B reduces Equation 4.36 to

$$B_{2n-1}(x) = -\zeta_o^{+2} \int_o^x R_{2n-1}(x) p_{2n-1}(\zeta_o x)dx \qquad (4.39)$$

In a similar manner it can be shown that, at x=1,

$$0 = A_{2n-1}(1)p_{2n-1}(\zeta_o) + B_{2n-1}(1)q_{2n-1}(\zeta_o) \qquad (4.40)$$

Hence, $A_{2n-1}(1)$ is

$$A_{2n-1}(1) = -\frac{q_{2n-1}(\zeta_o)}{p_{2n-1}(\zeta_o)} B_{2n-1}(1) \qquad (4.41)$$

Use of Equation 4.35 is now complete; attention is now directed at Equation 4.34 which is integrated to be

$$A_{2n-1}(x) - A_{2n-1}(1) = \zeta_o^{+2} \int_1^x R_{2n-1}(x) q_{2n-1}(\zeta_o x)dx \quad (4.42)$$

The $A_{2n-1}(1)$ on the left hand side of this equation has been found earlier in Equation 4.41. Hence, Equation 4.42 can be rewritten to be

$$A_{2n-1}(x) = -\frac{q_{2n-1}(\zeta_o)}{p_{2n-1}(\zeta_o)} B_{2n-1}(1) + \zeta_o^2 \int_1^x R_{2n-1}(x) q_{2n-1}(\zeta_o x)dx \qquad (4.43)$$

Equations 4.39 and 4.43 are to be used in Equation 4.31 to find $\overline{\theta}_{2n-1}(x)$, which is, in turn, operated by an inverse Legendre transform to get

$$\theta_1(x,\mu) = \sum_n (4n-1)[A_{2n-1}(x)p_{2n-1}(\zeta_o x) + B_{2n-1}(x)q_{2n-1}(\zeta_o x)]P_{2n-1}(\mu) \tag{4.44}$$

Equation 4.44 is the desired first order solution to the heat transfer problem.

The solution method presented above can be repeated for higher order solutions. However, because of the great complexity involved in obtaining the first order solution, those higher order solutions were not attempted in this work. Instead, those constants A and B in Equation 4.44 are re-examined to make the first order solution complete.

Equations 4.39 and 4.43 can be expressed in terms of g functions with the substitution of Equation 4.29 as follows

$$B_{2n-1}(x) = \frac{-4}{(2\cot^{-1}\zeta_o-\pi)}\frac{n(2n-1)}{(4n-1)}\int_o^x \frac{g_{2n-1}(x)}{(x^2+\zeta_o^{-2})} P_{2n-1}(\zeta_o x)dx \tag{4.45}$$

and

$$A_{2n-1}(x) = -\frac{q_{2n-1}(\zeta_o)}{p_{2n-1}(\zeta_o)} B_{2n-1}(1) + \frac{4}{(2\cot^{-1}\zeta_o-\pi)}\frac{n(2n-1)}{(4n-1)}$$

$$\int_1^x \frac{g_{2n-1}(x)}{(x^2+\zeta_o^{-2})} q_{2n-1}(\zeta_o x)dx \tag{4.46}$$

The appearance of $B_{2n-1}(1)$ on the right hand side of Equation 4.46 implies that $B_{2n-1}(x)$ must be found first before $A_{2n-1}(x)$ can be evaluated. Those g functions in the integrands have been found earlier for the fluid flow problem, where as the p and q functions can be expanded in series [36]. The integrals in Equation 4.45 and 4.46 can thus be carried out with a numerical integration.

CHAPTER V

RESULTS AND DISCUSSION

Data for g functions were computed using Equation 3.47 as shown in Figure 7, all g functions are of the same order. They range from 0.5 to 2.5. Recall that the magnitudes and slope of the g functions at x=0 are governed by the boundary conditions listed in Table 1. Since these are curves for the same Re number, the similarity of the g curves of Figure 7 is not unexpected.

Recall that the conditions for g_3 and g_1 at x=0 have been established in Chapter III, as $g_{3,1}=(1/9)g_{1,1}$, and inferrence was drawn at that time that the g functions should converge rapidly. This point is now substantiated by the curves shown in Figure 7. In fact, $g_5(x)$ for $\zeta_o=2.5$ is found to be everywhere zero. Finally, there appears to be little change in the trends evidenced by the g functions when the Re number is changed from 0.1 to 0.3, see Figure 8.

Stream Functions

The g functions permit the evaluation of the stream function, Equation 3.15. A plot of Ψ for Re=.1 and $\zeta_o=0.5$, 2, and 3 is presented in Figure 9. Notice that Re has been defined in this work as $2\bar{V}d/\nu$, and $\zeta_o=c/d$. d is the radius of the feeder tube, whereas c is the depth of the cap. A large ζ_o thus refers to an end cap having a large annular passage

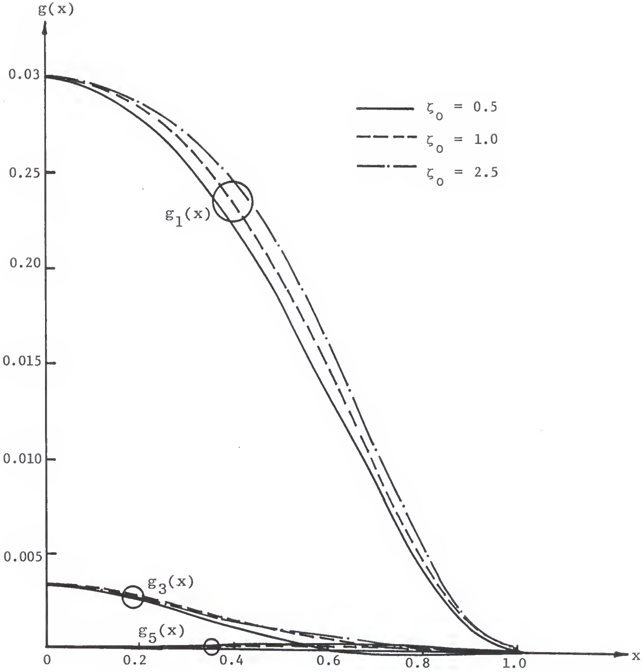


Figure 7. g_{2n-1} Curves for Re=0.1

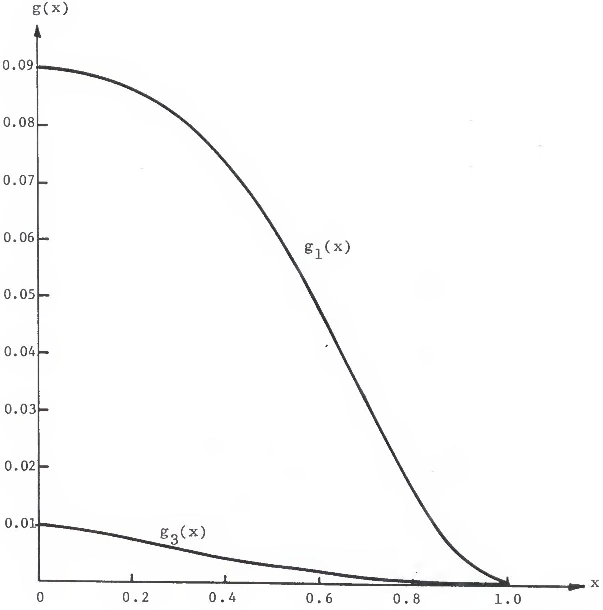


Figure 8. $g_{2n-1}(x)$ Curves for ζ_o=2.5 and Re=0.3

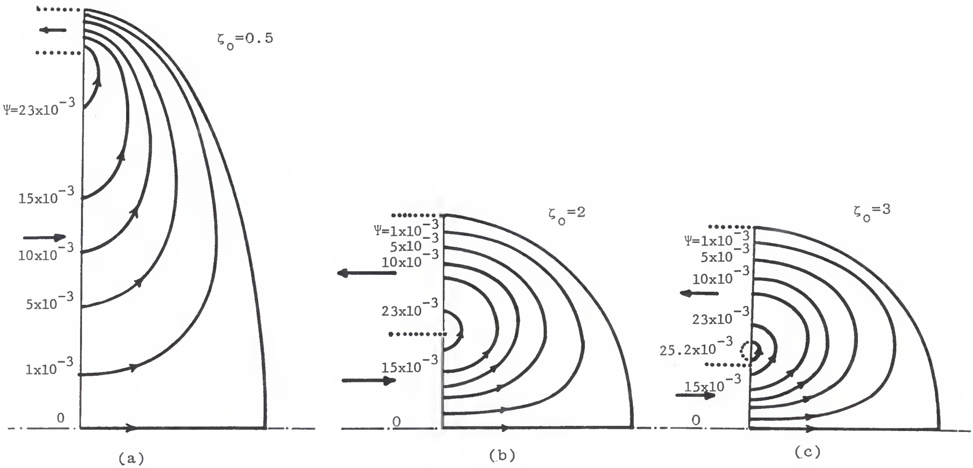


Figure 9. A Plot of the Streamline for Re=0.1

relative to the feeder tube. Such a condition is conducive to a circulation flow near the rim of the tube, and the stream lines in Figure 9(c) and 10 substantiate this point. A magnification of the flow pattern near the rim of the feeder tube shown in Figure 10(b) is provided in Figure 11.

Velocity Fields

The velocity field was determined using Equation 3.5. The velocity distribution was plotted in three dimensions and shown in two views for ζ_o=2 and Re=0.1 in Figure 12. At first sight, the entrance velocity in the figure may appear to be high compared with that at the exit. This turns out to be a matter of geometry. For the end cap under investigation in this plot, ζ_o=2=(c/d) is equivalent to an (a/d) ratio of 2.236; the exit velocity from the end cap is thus very small in order to satisfy continuity of flow. Also notice that this ζ_o=2 is close to the limit beyond which a circulation will appear near the rim of the tube.

Pressure Fields

The pressure drop in the end cap is expected to be large because the flow is deflected 180 degrees inside the cap. It is informative to examine how the pressure distributes in the flow region.

The pressure field was determined using Equation 3.67 and plotted in three dimensions for ζ_o=2 and Re=0.1, see Figure 13. The pressure drop in found to be quite marked. There is a large

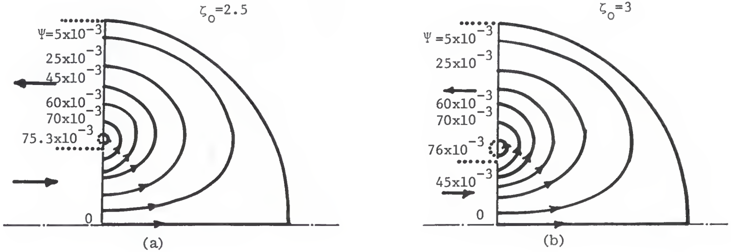


Figure 10. A Plot of the Streamlines for Re=0.3

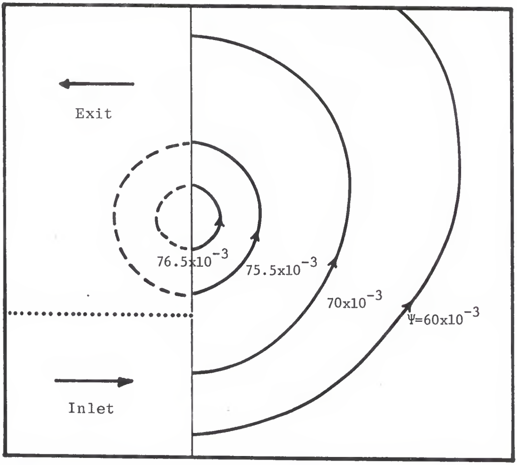


Figure 11. A Magnification of the Streamlines Near the Edge of the Feeder Tube for ζ_o=3 and Re=0.3

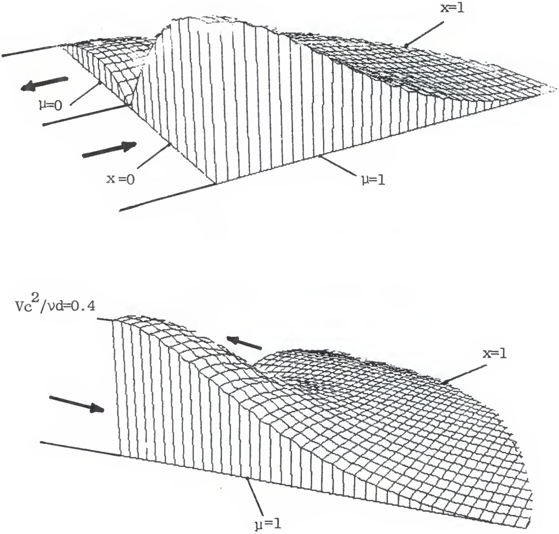


Figure 12. Two Views of the Speed Distribution in the End Cap for ζ_o=2 and Re=0.1

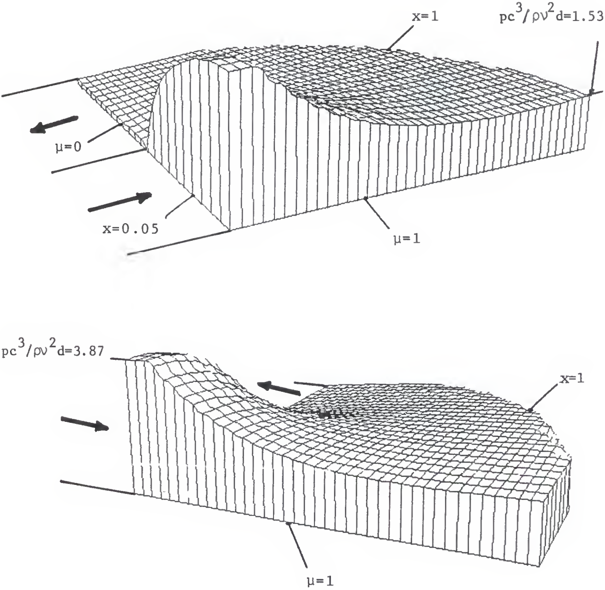


Figure 13. Two Views of the Pressure Distribution in the End Cap for $\zeta_o=2$ and Re=0.1

pressure exerted on the surface of the end cap in the axial direction. This is certainly a result of the conversion of the velocity head to pressure head as the axial velocity decelerates to zero when the flow impinges on the cap surface. The velocity drop in the axial direction has also been computed and tabulated as $uc^2/\nu d$ in Table 3.

Temperature Fields

Equation 4.10 was used to plot the temperature field in Figure 14. Conditions chosen for this plot are $\zeta_o=2$ and Re=0.1. The solid line in the figure represents the pure conduction case, that is, the zeroth order solution of the temperature distribution in the perturbation method. An addition of convection effect displaces the isotherms to the right, which is not unexpected. Notice that for the problem solved, the temperature is truncated to the first order; yet the conduction effect is dominant as is clearly seen in the plot.

Local Convective Coefficient

At the surface of the cap, the heat transfer coefficient is defined as $h(T_w-T_s)+k\left.\frac{\partial T}{\partial n}\right|_s = 0$. In terms of the Nusselt number Nu and the dimensionless temperature θ, the above definition becomes, $Nu = \left.\frac{\partial\theta}{\partial x}\right|_{x=1}$.

Equations 4.22 and 4.44 were used to determine the $Nu(\mu)$ plotted in Figure 15 for Pr=0.5, 0.6, 0.8, 0.9. The conditions are given in the legend. Notice that the Nusselt number decreases with and approaches the limiting case of a pure conductive situation of Nu=0.362. The Nusselt number is high

Table 3. A tabulation of axial velocity expressed in $uc^2/\nu d$ for Re=0.1

x \ ζ_o	0.5	1	1.5	2	2.5	3
0	$0.2500\text{x}10^{-1}$	0.1000	0.2250	0.4000	0.6250	0.9000
0.1	$0.2423\text{x}10^{-1}$	$0.9623\text{x}10^{-1}$	0.2135	0.3709	0.5620	0.7805
0.2	$0.2216\text{x}10^{-1}$	$0.8943\text{x}10^{-1}$	0.1856	0.3067	0.4377	0.5703
0.3	$0.1916\text{x}10^{-1}$	$0.7278\text{x}10^{-1}$	0.1499	0.2337	0.3138	0.3860
0.4	$0.1558\text{x}10^{-1}$	$0.5732\text{x}10^{-1}$	0.1128	0.1665	0.2122	0.2503
0.5	$0.1178\text{x}10^{-1}$	$0.4181\text{x}10^{-1}$	$0.7875\text{x}10^{-1}$	0.1108	0.1357	0.1553
0.6	$0.8102\text{x}10^{-2}$	$0.2762\text{x}10^{-1}$	$0.4995\text{x}10^{-1}$	$0.6761\text{x}10^{-1}$	$0.8040\text{x}10^{-1}$	$0.9012\text{x}10^{-1}$
0.7	$0.4835\text{x}10^{-2}$	$0.1583\text{x}10^{-1}$	$0.2762\text{x}10^{-1}$	$0.3627\text{x}10^{-1}$	$0.4219\text{x}10^{-1}$	$0.4662\text{x}10^{-1}$
0.8	$0.2254\text{x}10^{-2}$	$0.7088\text{x}10^{-1}$	$0.1199\text{x}10^{-1}$	$0.1539\text{x}10^{-1}$	$0.1764\text{x}10^{-1}$	$0.1929\text{x}10^{-1}$
0.9	$0.5850\text{x}10^{-3}$	$0.1769\text{x}10^{-2}$	$0.2921\text{x}10^{-2}$	$0.3688\text{x}10^{-2}$	$0.4181\text{x}10^{-2}$	$0.4541\text{x}10^{-2}$

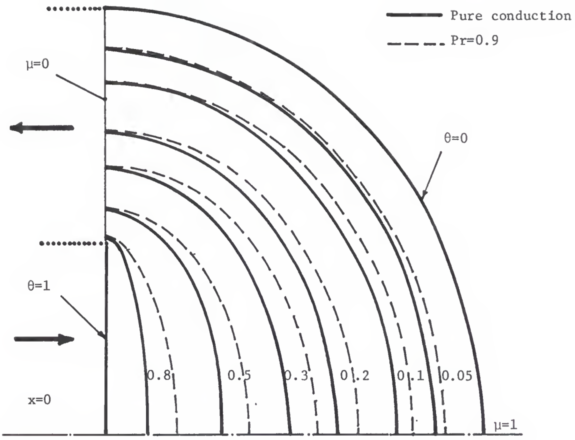


Figure 14. A Plot of Isotherms for $\zeta_o=2$ and Re=0.1

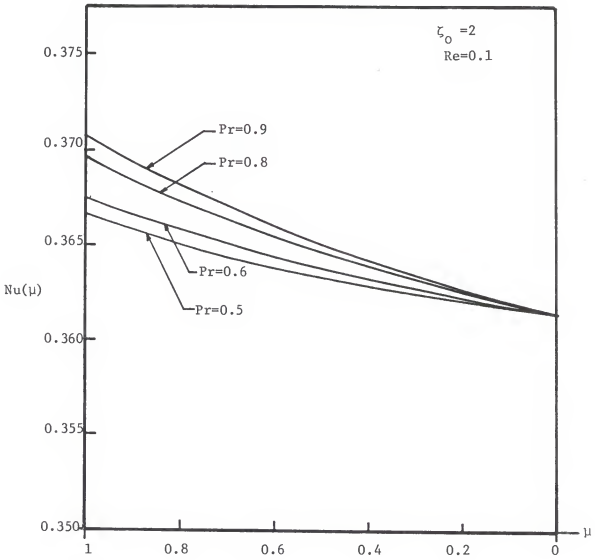


Figure 15. Nu(μ) Curves for $\zeta_o=2$ and Re=0.1

in the region where the wall pressure is high. This effect becomes less prominent if the Prandtl number is low.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

Based on the work done, one can draw the following conclusions:

1. An exact solution has been developed for fluid flow and heat transfer in the end cap of a single-pass, return flow heat exchanger for the special case of steady, low speed flow condition. In this case the product of velocity components can be neglected in the analysis. In the solution, the stream function is expressed in a series of Legendre polynomials in which only three terms are found to be significant. The series converges rapidly for the geometry and the Re number range considered in this study.

2. A parameter perturbation method is used to solve for the temperature filed in the end cap. The zeroth order perturbation solution corresponds to a pure conduction problem. Because of the great complexity involved in the solution, the perturbation is developed only up to the first order with a hypothetical case of uniform inlet and wall temperature conditions as boundary conditions.

3. In the fluid flow study, the obtained stream functions are used to determine the velocity and pressure fields. A recirculation flow is found near the rim of the feeder tube

when the tube annulus region has a cross section that is large compared to that of the tube.

4. For the heat transfer study, the conduction effect is found to be dominant. Larger convective coefficients are found in the wall region where the pressure is high.

Recommendations

Some recommendations are made as follows:

1. It is clear that, because of the great complexity involved in the exact solution, only some simplified cases can be studied. For those cases whose solutions are given in this work, further numerical solutions or experiments are needed for verification.

2. Whether a higher order perturbation is needed in the solution of the heat transfer problem is also subjected to a rigorous verification.

3. The range of the parameters (Re and Pr) studied in this work is quite limited. It is necessary to study their limits under which this analytical solution is useful. This can only be made by comparing the results obtained by numerical solutions of the problem.

APPENDICES

APPENDIX A

DERIVATION OF EQUATION (3.8)

It is convenient to establish relations between the cylindrical coordinates (r,z,ϕ) and the oblate spheroidal coordinates (η,θ,ϕ) with the help of the cartesian coordinates (x,y,z). In terms of the cylindrical coordinates, the cartesian coordinates can be expressed as [38]

$$x = \xi_1\xi_2$$

$$y = \xi_1 \sqrt{1-\xi_2^2}$$

$$z = \xi_3$$

where, $\xi_1=r$, $\xi_2=\cos\phi$ and $\xi_3=z$.

In terms of the oblate spheroidal coordinates (η,θ,ϕ), the cartesian coordinates become

$$x = \xi_3\sqrt{(\xi_1^2+d^2)(1-\xi_2^2)}$$

$$y = \sqrt{(\xi_1^2+d^2)(1-\xi_2^2)(1-\xi_3^2)}$$

$$z = \xi_1\xi_2$$

where, $\xi_1=d\sinh\eta$, $\xi_2=\cos\theta$ and $\xi_3=\cos\phi$.

The relations between the cylindrical and the spheroidal coordinates can then be established by relating these two sets of x,y and z equations. It follows that

$$r = d \cosh\eta \sin\theta$$

$$z = d \sinh\eta \cos\theta$$

By defining, $\cos\theta = \mu$ and $\sinh\eta = \zeta$, it is possible to rewrite r and z as

$$r = d(1+\zeta^2)^{\frac{1}{2}}(1-\mu^2)^{\frac{1}{2}} \qquad (A.1)$$

$$z = d\zeta\mu \qquad (A.2)$$

which relate (r,z) to (ζ,μ).

The introduction of the curvilinear coordinates developed above enables a speedy solution to the fluid flow problem. Such a transformation is not uncommon in the field of mathematical physics in the solution of punch, crack and torsion problems, among others (e.g., [39]).

It is easy to show that the surfaces μ=constant and ζ=constant are, respectively, hyperboloids of revolution and oblate spheroids. The spheroid $\zeta=0$ is the circular disk $r < d$ and $z=0$, while $\mu=0$ gives the circular ring surrounding the disk (Figure A.1).

In fluid mechanics the stream function ψ satisfies the continuity equation which can be expressed in vector notation as follows

$$\nabla \cdot V = 0 \qquad (A.3)$$

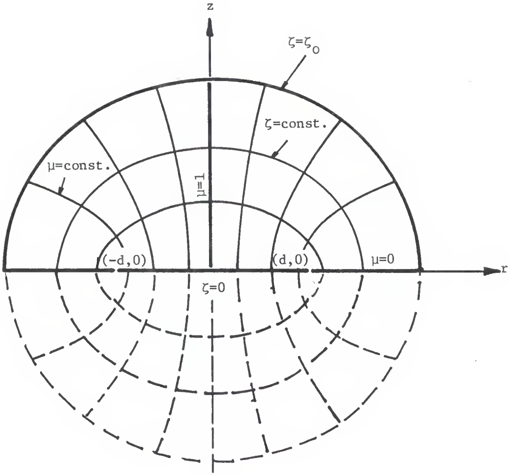


Figure A.1. Oblate Spheroidal Coordinate System
——— : Fluid Flow Region

Use of axisymmetry, the divergence of V may be written as

$$\nabla \cdot V = \frac{1}{h_1\ h_2\ h_3}\ [\frac{\partial}{\partial \xi_1}(h_2 h_3 V_1) + \frac{\partial}{\partial \xi_2}(h_3 h_1 V_2)] \qquad (A.4)$$

where h_α are scale factors, defined as

$$h_\alpha^2 = \sum_i \left(\frac{\partial x_i}{\partial \xi_\alpha}\right)^2 \ , \ \alpha=1,2,3$$

where x_i are the cartesian coordinates.

These factors take the following forms in the oblate spheroidal coordinates:

$$h_1 = \sqrt{\frac{\xi_1^2+d^2\xi_2^2}{\xi_1^2+d^2}}$$

$$h_2 = \sqrt{\frac{\xi_1^2+d^2\xi_2^2}{1-\xi_2^2}}$$

$$h_3 = \sqrt{\frac{(\xi_1^2+d^2)(1-\xi_2^2)}{1-\xi_3^2}}$$

Substituting h_α into Equation A.4 and making use of Equation A.3 gives

$$\frac{\partial}{\partial \xi_1}\ [(\xi_1^2+d^2\xi_2^2)^{\frac{1}{2}}(\xi_1^2+d^2)^{\frac{1}{2}}V_1] + \frac{\partial}{\partial \xi_2}\ [(\xi_1^2+d^2\xi_2^2)^{\frac{1}{2}}(1-\xi_2^2)^{\frac{1}{2}}V_2] = 0$$

Since $\xi_1 = d\zeta$ and $\xi_2 = \mu$, the above equation can be recast in terms of (ζ,μ) coordinates as

$$\frac{\partial}{\partial\zeta}\,[d^2(\zeta^2+\mu^2)^{\frac{1}{2}}(1+\zeta^2)^{\frac{1}{2}}u] + \frac{\partial}{\partial\mu}\,[d^2(\zeta^2+\mu^2)^{\frac{1}{2}}(1-\mu^2)^{\frac{1}{2}}v] = 0$$

It is now possible to define the stream function ψ, which is related to the velocity components u and v as follows

$$u = \frac{-1}{d^2(\zeta^2+\mu^2)^{\frac{1}{2}}(1+\zeta^2)^{\frac{1}{2}}}\,\frac{\partial\psi}{\partial\mu} \qquad (A.5)$$

$$v = \frac{1}{d^2(\zeta^2+\mu^2)^{\frac{1}{2}}(1-\mu^2)^{\frac{1}{2}}}\,\frac{\partial\psi}{\partial\zeta} \qquad (A.6)$$

Clearly, the continuity equation is satisfied by ψ identically. These velocity components have been used to write the velocity V in Equation 3.5.

The derivation given in Chapter III for the momentum equation involves use of the definition of vorticity given as

$$\omega = \nabla x V$$

In terms of the curvilinear coordinates (ξ_1,ξ_2,ξ_3), this ω becomes

$$\omega = \frac{1}{h_1h_2}\,[\frac{\partial}{\partial\xi_1}(h_2V_2) - \frac{\partial}{\partial\xi_2}(h_1V_1)]\hat{\delta}_3$$

In the (ζ,μ) two dimensional coordinate system that is of interest in this work,

$$\omega = \frac{(1-\mu^2)^{\frac{1}{2}}(1+\zeta^2)^{\frac{1}{2}}}{d(\zeta^2+\mu^2)}\left\{\frac{\partial}{\partial\zeta}\left[\frac{(\zeta^2+\mu^2)^{\frac{1}{2}}}{(1-\mu^2)^{\frac{1}{2}}}\,v\right] - \frac{\partial}{\partial\mu}\left[\frac{(\zeta^2+\mu^2)^{\frac{1}{2}}}{(1+\zeta^2)^{\frac{1}{2}}}\,u\right]\right\}\hat{\delta}_\phi$$

This ω equation can now be expressed in terms of the stream function as

$$\omega = \frac{(1-\mu^2)^{\frac{1}{2}}(1+\zeta^2)^{\frac{1}{2}}}{d^3(\zeta^2+\mu^2)} \left[\frac{1}{(1-\mu^2)} \frac{\partial^2\psi}{\partial\zeta^2} + \frac{1}{(1+\zeta^2)} \frac{\partial^2\psi}{\partial\mu^2} \right] \hat{\delta}_\phi \qquad (A.7)$$

Finally, in order to obtain an expression for the governing equation in (ζ,μ) coordinates, the curl of Equation A.7 is repeated twice, which gives

$$\nabla x \nabla x \omega = \frac{(1+\mu^2)}{d^2} \frac{\partial^2}{\partial\zeta^2} \left\{ \frac{1}{(\zeta^2+\mu^2)} \left[(1+\zeta^2) \frac{\partial^2\psi}{\partial\zeta^2} + (1-\mu^2) \frac{\partial^2\psi}{\partial\mu^2} \right] \right\}$$

$$+ \frac{(1-\mu^2)}{d^2} \frac{\partial^2}{\partial\mu^2} \left\{ \frac{1}{(\zeta^2+\mu^2)} \left[(1+\zeta^2) \frac{\partial^2\psi}{\partial\zeta^2} + (1-\mu^2) \frac{\partial^2\psi}{\partial\mu^2} \right] \right\} = 0 \qquad (A.8)$$

A close examination of this equation suggests the definition of an operator

$$D^2 \equiv (1+\zeta^2) \frac{\partial^2}{\partial\zeta^2} + (1-\mu^2) \frac{\partial^2}{\partial\mu^2}$$

which permits a rewriting of Equation A.8 as

$$D^2 \left[\frac{1}{(\zeta^2+\mu^2)} D^2\psi \right] = 0 \qquad (A.9)$$

An alternative expression to Equation A.9 is to consider use of dimensionless quantities Ψ and x,

$$\Psi = \frac{\psi}{\nu d}, \quad x = \frac{\zeta}{\zeta_o}$$

Equation A.9 is thus changed to

$$D^2 \left[\frac{1}{x^2+\mu^2\zeta_o^{-2}} D^2\Psi \right]=0 \qquad (A.10)$$

where,

$$D^2 \equiv (x^2+\zeta_o^{-2}) \frac{\partial^2}{\partial x^2} + (1-\mu^2) \frac{\partial^2}{\partial \mu^2} \qquad (A.11)$$

APPENDIX B

DERIVATION OF EQUATIONS 3.21 THROUGH 3.23

Let $f(\mu)$ in Equation 3.13 be a function that satisfies Dirichlet conditions in the interval (-1,1). Then, $f(\mu)$ can be expanded as

$$f(\mu) = (1-\mu^2) \sum_{n=1}^{\infty} \beta_{2n-1} P'_{2n-1}(\mu) \tag{B.1}$$

Invoking use of the orthogonality characteristics of $(1-\mu^2)P'_{2n-1}(\mu)$ over (-1,1), it can be shown that [33]

$$\beta_{2n-1} = \frac{4n-1}{4n(2n-1)} \int_{-1}^{1} f(\mu)\, P'_{2n-1}(\mu)\, d\mu \tag{B.2}$$

In the special case of a fully developed flow at the exit of the feed pipe, the velocity is related to the radius according to [3]

$$V_z = u = 2\overline{V}[1 - (\frac{r}{d})^2] \tag{B.3}$$

Since at the exit plane $\zeta=0$, Equation 3.3 can be used to rewrite the velocity u as a function of μ, that is,

$$u = 2\overline{V}[1-(1-\mu^2)]=2\overline{V}\mu^2 \tag{B.4}$$

In addition, u is related to ψ according to Equation A.5. At the location $\zeta=0$,

$$u = -\frac{1}{d^2\mu}\frac{\partial\psi}{\partial\mu} \tag{B.5}$$

Equations B.4 and B.5 can now be combined to give

$$-\frac{1}{d^2\mu}\frac{\partial\psi}{\partial\mu} = 2\overline{V}\mu^2 \tag{B.6}$$

and $\psi(0,\mu)$ derived as

$$\psi(0,\mu) = -\overline{V}\, d^2 \frac{\mu^4}{2} + c \tag{B.7}$$

The constant c of integration can be determined from the fact that, along the axis of symmetry (i.e., $\mu=1$), the stream function is zero. Hence,

$$\psi(0,\mu) = \frac{\overline{V}d^2}{2}(1-\mu^4) \tag{B.8}$$

In terms of Reynolds number $Re=2\overline{V}d/\nu$, this ψ can be changed to Ψ as

$$\Psi(0,\mu)=Re(1-\mu^4)/4 \tag{B.9}$$

At this stage it is possible to use the above equation in Equation B.1 to determine β_{2n-1}. With the help of Equation B.2, it can be deduced that

$$\beta_1 = 3/10\ \mathrm{Re}$$

$$\beta_3 = 1/30\ \mathrm{Re}$$

and

$$\beta_{2n-1}=0 \qquad \text{for } n \geq 3 \qquad \text{(B.10)}$$

Substituting these in Equation B.1 gives

$$\Psi(0,\mu)=f(\mu)=\tfrac{1}{2}\mathrm{Re}(1-\mu^2)[3/5\ P_1'(\mu)+1/15\ P_3'(\mu)] \qquad \text{(B.11)}$$

Comparing Equation B.11 with Equation 3.15 enables the g function to be found immediately as

$$g_1(0) = 3/10\ \mathrm{Re}$$

$$g_2(0) = 1/30\ \mathrm{Re}$$

and

$$g_{2n-1}(0)=0 \qquad \text{for } n \geq 3 \qquad \text{(B.12)}$$

APPENDIX C

DERIVATION OF EQUATION 3.36

The governing equation to be solved for the fluid flow problem is

$$D^2 \frac{1}{x^2+\mu^2\zeta_o^{-2}} D^2\Psi = 0 \qquad (C.1)$$

where,

$$D^2 \equiv (x^2+\zeta_o^{-2}) \frac{\partial^2}{\partial x^2} + (1-\mu^2) \frac{\partial^2}{\partial \mu^2}$$

A solution Ψ has been suggested in the form

$$\Psi = (1-\mu^2) \sum_{n=1}^{\infty} g_{2n-1}(x) P'_{2n-1}(\mu) \qquad (C.2)$$

By using the identity

$$[(1-\mu^2)P'_{2n-1}(\mu)]'' + 2n(2n-1)P'_{2n-1}(\mu) = 0 \qquad (C.3)$$

and defining

$$G_{2n-1}(x) = (x^2+\zeta_o^{-2})g''_{2n-1}(x) - 2n(2n-1)g_{2n-1}(x) \qquad (C.4)$$

it can be derived that

$$D^2\Psi = (1-\mu^2) \sum_{n=1}^{\infty} G_{2n-1}(x)P'_{2n-1}(\mu) \qquad (C.5)$$

Equation 3.24 can be derived by subsituting Equation C.5 into Equation C.1 and, after some manipulation, this gives

$$\begin{aligned}
&\sum_{n=1}^{\infty} (x^2+\zeta_o^{-2})[(x^2+\mu^2\zeta_o^{-2})G''_{2n-1}(x)-4xG'_{2n-1}(x)]\, P'_{2n-1}(\mu) \\
&- \left\{ 4\zeta_o^{-2}\mu[\underline{(1-\mu^2)P'_{2n-1}(\mu)}]' + 2[(n+1)(2n-3)(x^2+\underline{\mu^2\zeta_o^{-2}})+6\underline{\mu^2\zeta_o}^{-2}-2\zeta_o^{-2}] \right. \\
&\left. \underline{P'_{2n-1}(\mu)} \right\} G_{2n-1}(x) = 0 \qquad \text{(C.6)}
\end{aligned}$$

In order to sort out G from this equation, the identities 3.26 and 3.27 are substituted for those underlined terms in Equation C.6, and after some algebra, Equation C.6 is transformed to

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left\{ (x^2+\zeta_o^{-2}) \left[x^2+\zeta_o^{-2} \frac{8n^2-4n-3}{(4n+1)(4n-3)} \right] G''_{2n-1} -4x(x^2+\zeta_o^{-2})G'_{2n-1} \right. \\
&+ \left\{ \frac{8n(2n-1)-2(2n^2-n+3)(8n^2-4n-3)}{(4n+1)(4n-3)} \zeta_o^{-2} -2\left[(n+1)(2n-3)x^2-2\zeta^{-2} \right] \right\} \\
&\left. G_{2n-1} \right\} P'_{2n-1}(\mu) + \left\{ (x^2+\zeta_o^{-2}) \frac{2n(2n-1)}{(4n-1)(4n-3)} \zeta_o^{-2}\, G''_{2n-1} \right. \\
&\left. - \left[\frac{4n(2n-1)(2n+1(n+1)}{(4n-1)(4n-3)} \zeta_o^{-2} \right] G_{2n-1} \right\} P'_{2n-3(\mu)} \\
&+ \left\{ (x^2+\zeta_o^{-2}) \frac{2n(2n-1)}{(4n-1)(4n+1)} \zeta_o^{-2}\, G_{2n-1} - \left[\frac{4n(2n-1(n-1)(2n-3)}{(4n-1)(4n+1)} \zeta_o^{-2} \right] \right. \\
&\left. G_{2n-1} \right\} P'_{2n+1}(\mu) = 0 \qquad \text{(C.7)}
\end{aligned}$$

A general equation for G can then be derived from Equation C.7 by (i) expanding the series out, (ii) grouping terms of $P'_1(\mu)$, $P'_3(\mu)$, ... , and finally (iii) equating each coefficient to zero. The differential equation for G is given as Equation 3.28.

Following the procedure described in the text to derive Equation 3.36, it can be shown that the coefficients A to Q in this equation can be related to a, b, and c (Equations 3.29 through 3.35) by

$$A_{2n-1}=(x^2+\zeta_o^{-2})b^{(2)}_{2n-1} \tag{C.8}$$

$$B_{2n-1}=4xb^{(2)}_{2n-1} + (x^2+\zeta_o^{-2})b^{(1)}_{2n-1} \tag{C.9}$$

$$C_{2n-1}=-2n(n-1)(2n+1)b^{(2)}_{2n-1} + 2xb^{(1)}_{2n-1} + (x^2+\zeta_o^{-2})b^{(0)}_{2n-1} \tag{C.10}$$

$$D_{2n-1}=-2n(2n-1)b^{(1)}_{2n-1} \tag{C.11}$$

$$E_{2n-1}=-2n(2n-1)b^{(0)}_{2n-1} \tag{C.12}$$

$$H_{2n-1}=(x^2+\zeta_o^{-2})a^{(2)}_{2n-1} \tag{C.13}$$

$$I_{2n-1}=4xa^{(2)}_{2n-3} \tag{C.14}$$

$$J_{2n-1}=-2(n-2)(2n-1)a^{(2)}_{2n-3} + (x^2+\zeta_o^{-2})a^{(0)}_{2n-1} \tag{C.15}$$

$$K_{2n-1}=-2(n-1)(2n-3)a^{(0)}_{2n-3} \tag{C.16}$$

$$L_{2n-1}=(x^2+\zeta_o^{-2})c^{(2)}_{2n+1} \tag{C.17}$$

$$M_{2n-1}=4xc^{(2)}_{2n+1} \tag{C.18}$$

$$N_{2n-1}=-2n(2n+3)c^{(2)}_{2n+1} + (x^2+\zeta_o^{-2})c^{(0)}_{2n+1} \tag{C.19}$$

$$Q_{2n-1}=-2(n+1)(2n+1)c^{(0)}_{2n+1} \tag{C.20}$$

APPENDIX D

SOLUTION OF EQUATION 3.36 BY A FINITE DIFFERENCE APPROACH

The set of equations to be solved for g is

$$A_{2n-1}g^{iv}_{2n-1} + B_{2n-1}g'''_{2n-1} + C_{2n-1}g''_{2n-1} + D_{2n-1}g'_{2n-1} + E_{2n-1}g_{2n-1}$$

$$= -R_{2n-1} - S_{2n-1} \quad \text{for } n=1,2, \ldots , N \qquad (D.1)$$

$$R_1=0$$

$$R_{2n-1}=H_{2n-1}g^{iv}_{2n-1} + I_{2n-1}g'''_{2n-3} + J_{2n-1}g''_{2n-3} + K_{2n-1}g_{2n-3}$$

$$\text{for } n=2,3, \ldots , N \qquad (D.2)$$

$$S_{2n-1}=L_{2n-1}g^{iv}_{2n+1} + M_{2n-1}g'''_{2n+1} + N_{2n-1}g''_{2n+1} + Q_{2n-1}g_{2n+1}$$

$$\text{for } n=1,2, \ldots , N-1 \qquad (D.3)$$

$$S_{2n-1}=0$$

Coefficients A through Q in these equations have been given in Appendix C.

Using the central difference scheme given in [40] permits writing the derivatives of g as

$$g'(x_j)= \frac{1}{2h}(-g(x_{j-1})+g(x_{j+1}))$$

$$g''(x_j)=\frac{1}{h^2}(g(x_{j-1})-2g(x_j)+g(x_{j+1}))$$

$$g'''(x_j) = \frac{1}{2h^3}(-g(x_{j-2})+2g(x_{j-1})-2g(x_{j+1})+g(x_{j+2}))$$

$$g^{iv}(x_j) = \frac{1}{h^4}(g(x_{j-2})-4g(x_{j-1})+6g(x_j)-4g(x_{j+1})+g(x_{j+2}))$$

where x_j denotes position at index j, and h represents the step size in x-direction. Thus,

$$x_j=(j-1)h$$

The derivatives of g given above are accurate to h^2 according to Taylor's expansion. The relation between x and j is shown in Figure D.1.

Equations D.1 through D.3 will be used to write g relations at point x_j. It is thus convenient to use double subscripts, the first one identifying the term in the series expansion in Equation 3.15, the second one, the position of the point, for example,

$$g_{2n-1}(x_j)=g_{2n-1,j}$$

Substituting Equation D.4 into Equation D.1 gives

$$\begin{aligned}&(A_{2n-1,j}-B_{2n-1,j}\ \frac{h}{2})g_{2n-1,j-2}+(-4A_{2n-1,j}+B_{2n-1,j}\ h+C_{2n-1,j}\ h^2\\&-D_{2n-1,j}\ \frac{h^3}{2})g_{2n-1,j-1}+(6A_{2n-1,j}-2C_{2n-1,j}\ h^2+E_{2n-1,j}\ h^4)g_{2n-1,j}\\&+(-4A_{2n-1,j}-B_{2n-1,j}\ h+C_{2n-1,j}\ h^2+D_{2n-1,j}\ \frac{h^3}{2})g_{2n-1,j+1}\\&+(A_{2n-1,j}+B_{2n-1,j}\ \frac{h}{2})g_{2n-1,j+2} = -R_{2n-1,j}\ h^4-S_{2n-1,j}\ h^4 \qquad (D.5)\end{aligned}$$

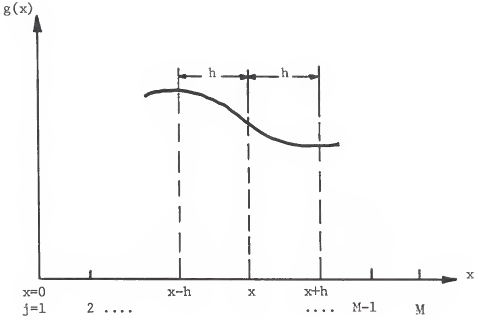


Figure D.1. g(x) in the Neighborhood of a Point x and the Discretization Scheme

Likewise, Equation D.2 is changed to

$$\begin{aligned}&(H_{2n-1,j}-I_{2n-1,j}\,\tfrac{h}{2})g_{2n-3,j-2}+(-4H_{2n-1,j}+I_{2n-1,j}\,h+J_{2n-1,j}h^{2})\\&g_{2n-3,j-1}+(6H_{2n-1,j}-2J_{2n-1,j}\,h^{2}+K_{2n-1,j}\,h^{4})g_{2n-3,j}\\&+(-4H_{2n-1,j}-I_{2n-1,j}\,h+J_{2n-1,j}\,h^{2})g_{2n-3,j+1}+(H_{2n-1,j}+I_{2n-1,j}\,\tfrac{h}{2})\\&g_{2n-3,j+2}=R_{2n-1,j}\,h^{4}\qquad\qquad (D.6)\end{aligned}$$

and for Equation D.3,

$$\begin{aligned}&(L_{2n-1,j}-M_{2n-1,j}\,\tfrac{h}{2})g_{2n+1,j-2}+(-4L_{2n-1,j}+M_{2n-1,j}\,h+N_{2n-1,j}\,h^{2})\\&g_{2n+1,j-1}+(6L_{2n-1,j}-2N_{2n-1,j}\,h^{2}+Q_{2n-1,j}\,h^{4})g_{2n+1,j}\\&+(-4L_{2n-1,j}-M_{2n-1,j}\,h+N_{2n-1,j}\,h^{2})g_{2n+1,j+1}+(L_{2n-1,j}+M_{2n-1,j}\,\tfrac{h}{2})\\&g_{2n+1,j+2}=S_{2n-1,j}\,h^{4}\qquad\qquad (D.7)\end{aligned}$$

The boundary conditions $g'(0)=0$ and $g'(1)=0$ can be written as

$$g_{2n-1,0}=g_{2n-1,2} \quad \text{and} \quad g_{2n-1,M-1}=g_{2n-1,M+1} \qquad (D.8)$$

respectively, and $n=1,2,3, \ldots , N$.

By taking j to be 2 through M-1 and making use of Equation D.8 it is possible to write Equation D.5 in a matrix for the nth term in the series expansion 3.15. That is,

$$\begin{bmatrix} 0 & 0\ldots & & & & & 0 \\ T_{21} & T_{22} & T_{23} & T_{24} & 0.. & & 0 \\ T_{31} & T_{32} & T_{33} & T_{34} & T_{35} & 0.. & 0 \\ 0 & T_{M-2\ M-4}\cdot\cdot & \cdots & & & T_{M-2\ M} & \\ \cdot & & T_{M-1\ M-3}\cdots & & & T_{M-1\ M} & \\ 0 & 0\ldots & 0 & & & 0 & 0 \end{bmatrix} \begin{Bmatrix} g_{2n-1,1} \\ g_{2n-1,2} \\ g_{2n-1,3} \\ \vdots \\ g_{2n-1,M-1} \\ g_{2n-1,M} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -R_{2n-1,2}h^4-S_{2n-1,2}h^4 \\ \cdot \\ \cdot \\ -R_{2n-1,M-1}h^4-S_{2n-1,M-1}h^4 \\ 0 \end{Bmatrix} \tag{D.9}$$

which is valid for n=1,2, ... , N. The elements in the coefficient matrix are given as follows

$$T_{22}=7A_{2n-1,2}-B_{2n-1,2}\,\frac{h}{2}-2C_{2n-1,2}\,h^2+E_{2n-1,2}\,h^4$$

$$T_{M-1\ M-1}=7A_{2n-1,M-1}+B_{2n-1,M-1}\,\frac{h}{2}-2C_{2n-1,M-1}\,h^2+E_{2n-1,M-1}\,h^4$$

$$T_{33}=\cdots=T_{M-2\ M-2}=6A_{2n-1,j}-2C_{2n-1,j}h^2+E_{2n-1,j}h^4\ ;\ j=3,\ ..\ ,\ M-2$$

$$T_{21}=\cdots=T_{M-1\ M-2}=-4A_{2n-1,j}+B_{2n-1,j}h+C_{2n-1,j}h^2+D_{2n-1,j}\,\frac{h^3}{2};$$

$$j=2,\ ..\ ,\ M-1$$

$$T_{23}=\cdots=T_{M-1\ M}=-4A_{2n-1,j}-B_{2n-1,j}h+C_{2n-1,j}h^2+D_{2n-1,j}\,\frac{h^3}{2};$$

$$j=2,\ ..\ ,\ M-1$$

$$T_{31}=\cdots=T_{M-1\ M-3}=A_{2n-1,j}-B_{2n-1,j}\,\frac{h}{2};\quad j=3,\ \ldots.\ ,\ M-1$$

$$T_{24}=\cdots=T_{M-2\ M}=A_{2n-1,j}+B_{2n-1,j}\,\frac{h}{2};\quad j=2,\ \ldots\ ,\ M-2 \tag{D.10}$$

In a like manner, D.6 can be expressed as

$$\begin{bmatrix} 0 & 0 .. & & & & & .. \; 0 \\ R_{21} & R_{22} & R_{23} & R_{24} & 0 \; .. & & 0 \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} & 0 .. & 0 \\ & R_{M-2\ M-4} & \cdots & & & & R_{M-2\ M} \\ & & & R_{M-1\ M-3} & & .. & R_{M-1\ M} \\ 0 & .. & & 0 \; .. & & & 0 \end{bmatrix} \begin{Bmatrix} g_{2n-3,1} \\ . \\ . \\ . \\ . \\ g_{2n-3,M} \end{Bmatrix} = \begin{Bmatrix} 0 \\ R_{2n-1,2}h^4 \\ \vdots \\ \vdots \\ R_{2n-1,M-1}h^4 \\ 0 \end{Bmatrix}$$

(D.11)

for n=2,3, ... , N. The elements in the coefficient matrix are

$$R_{22}=7H_{2n-1,2}-I_{2n-1,2}h-2J_{2n-1,2}h^2+K_{2n-1,2}h^4$$

$$R_{M-1,M-1}=7H_{2n-1,M-1}+I_{2n-1,M-1}\frac{h}{2}\text{-}2J_{2n-1,M-1}\ h^2+K_{2n-1,M-1}h^4$$

$$R_{33}= \cdots = R_{M-2\ M-2}=6H_{2n-1,j}-2J_{2n-1,j}h^2+K_{2n-1,j}h^4 \ ; \ j=3,..,M-2$$

$$R_{21}= \cdots = R_{M-1\ M-2}= -4H_{2n-1,j}+I_{2n-1,j}h+J_{2n-1,j}h^2 \ ; \ j=2,..,M-1$$

$$R_{23}= \cdots = R_{M-1\ M}= -4H_{2n-1,j}-I_{2n-1,j}h+J_{2n-1,j}h^2 \ ; \quad j=2,..,M-1$$

$$R_{31}= \cdots = R_{M-1\ M-3}=H_{2n-1,j}-I_{2n-1,j}\ \frac{h}{2} \ ; \quad j=3, \ ... \ ,M-1$$

$$R_{24}= \cdots = R_{M-2\ M}=H_{2n-1,j}+I_{2n-1,j}\ \frac{h}{2} \ ; \ j=2, \ ... \ ,M-2 \qquad \text{(D.12)}$$

Finally, Equation D.7 is expressed as

$$\begin{bmatrix} 0\ . & 0.. & & & & & 0 \\ S_{21} & S_{22} & S_{23} & S_{24} & 0.. & & 0 \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & 0.. & 0 \\ \vdots & & S_{M-2\ M-4} & \cdots & & & S_{M-2\ M} \\ 0.. & & & S_{M-1\ M-3} & \cdot\cdot & & S_{M-1\ M} \\ 0... & & & 0.. & & & 0 \end{bmatrix} \begin{Bmatrix} g_{2n+1,1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \vdots \\ g_{2n+1,M} \end{Bmatrix} = \begin{Bmatrix} 0 \\ S_{2n-1,2}h^4 \\ \vdots \\ \vdots \\ S_{2n-1,M-1}h^4 \\ 0 \end{Bmatrix} \tag{D.13}$$

for n=1,2, ... ,N-1. Elements S are

$$S_{22}=7L_{2n-1,2}-M_{2n-1,2}\ \frac{h}{2}-2N_{2n-1,2}h^2+Q_{2n-1,2}h^4$$

$$S_{M-1,M-1}=7L_{2n-1,M-1}+M_{2n-1,M-1}\ \frac{h}{2}-2N_{2n-1,M-1}h^2+Q_{2n-1,M-1}h^4$$

$$S_{33}=\ \dots\ =S_{M-2\ M-2}=6L_{2n-1,j}-2N_{2n-1,j}h^2+Q_{2n-1,j}h^4\ \ ;\ j=3,..,M-2$$

$$S_{21}=\ \dots\ =S_{M-1\ M-2}=-4L_{2n-1,j}+M_{2n-1,j}h+N_{2n-1,j}h^2\ \ ;\ j=2,\dots,M-1$$

$$S_{23}=\ \dots\ =S_{M-1\ m}=-4L_{2n-1,j}-M_{2n-1,j}h+N_{2n-1,j}h^2\ \ ;\ j=2,\dots,M-1$$

$$S_{31}=\ \dots\ =S_{M-1\ M-3}=L_{2n-1,j}-M_{2n-1,j}\ \frac{h}{2}\ \ ;\ j=3,..,M-1$$

$$S_{24}=\ \dots\ =S_{M-2\ M}=L_{2n-1,j}+M_{2n-1,j}\ \frac{h}{2}\ \ ;\ j=2,\dots,M-2 \tag{D.14}$$

Those three matrix equations given above can now be combined to give

$$\begin{bmatrix} [\underline{T}]_{n=1} & [\underline{S}]_{n=1} & 0.. & 0 \\ [\underline{R}]_{n=2} & [\underline{T}]_{n=2} & [\underline{S}]_{n=2} & 0 \\ & & & [\underline{S}]_{n=N-1} \\ 0 & 0 & [\underline{R}]_{n=N} & [\underline{T}]_{n=N} \end{bmatrix} \begin{Bmatrix} \underline{\Gamma}_1 \\ \underline{\Gamma}_3 \\ \vdots \\ \underline{\Gamma}_{2N-3} \\ \underline{\Gamma}_{2n-1} \end{Bmatrix} = 0 \qquad \text{(D.15)}$$

where,

$$\underline{\Gamma}_1 = \begin{Bmatrix} g_{1,1} \\ g_{1,2} \\ \vdots \\ g_{1,M} \end{Bmatrix} \text{ (D.16) and } \underline{\Gamma}_3 = \begin{Bmatrix} g_{3,1} \\ g_{3,2} \\ \vdots \\ g_{3,M} \end{Bmatrix} \qquad \text{(D.17)}$$

and so on . . .

Components g given in $\underline{\Gamma}$ vectors can be further simplified by using those boundary conditions which have not been used so far. Thus,

$g_1(0)=3/10$ Re is changed to $g_{1,1}=3/10$ Re (D.18)

$g_3(0)=1/30$ Re becomes $g_{3,1}=1/30$ Re (D.19)

$g_{2n-1}(0)=0$ for $n \geq 3$ becomes $g_{5,1}=g_{7,1}=..=0$ (D.20)

$g_{2n-1}(1)=0$ becomes $g_{1,M}=g_{3,M}=..=0$ (D.21)

The computer program is compiled in Appendix H.

APPENDIX E

DERIVATION OF EQUATIONS 3.50 AND 3.51

Equations A.1 and A.2 in Appendix A can be used to derive ζ, μ as functions of r and z as follows

$$\zeta = \frac{1}{2^{\frac{1}{2}}d^2} \{(r^2d^2+d^2z^2-d^4)+ \sqrt{(r^2d^2+d^2z^2-d^4)^2+4d^6z^2}\}^{\frac{1}{2}} \qquad (E.1)$$

$$\mu = \frac{z}{d\zeta} = 2^{\frac{1}{2}}dz \ \{(r^2d^2+d^2z^2-d^4) + \sqrt{(r^2d^2+d^2z^2-d^4)^4+4d^6z^2}\}^{\frac{1}{2}} \qquad (E.2)$$

Consider cylindrical coordinates (r,z) for a moment. These coordinates can be related to R and α based on the geometry given in Figure 6. Hence,

$$r=d+R\sin\alpha \qquad (E.3)$$

$$z=R \cos\alpha \qquad (E.4)$$

Substituting Equations E.3 and E.4 into Equations E.1 and E.2 gives

$$\zeta =(\tfrac{R}{2})^{\frac{1}{2}} \ \frac{1}{d} \ [(R+2d \sin\alpha)+(R^2+4d^2+4d \ R\sin\alpha)^{\frac{1}{2}}]^{\frac{1}{2}} \qquad (E.5)$$

$$\mu =(2R)^{\frac{1}{2}} \cos\alpha[(R+2d \sin\alpha)+(R^2+4d^2+4d \ R\sin\alpha)^{\frac{1}{2}}]^{-\frac{1}{2}} \qquad (E.6)$$

The expressions inside the braces in these two equations are now designated as $\Lambda_1(R)$ (for ζ) and $\Lambda_2(R)$ (for μ), and Taylor's expansions are used to write

$$\zeta = \left(\frac{R}{2}\right)^{\frac{1}{2}} \frac{1}{d} \left[\Lambda_1(0) + \frac{\Lambda_1^{'}(0)}{1!} R + \frac{\Lambda_1^{''}(0)}{2!} R^2 + \ldots\right] \qquad (E.7)$$

$$\mu = (2R)^{\frac{1}{2}} \cos\alpha\left[\Lambda_2(0) + \frac{\Lambda_2^{'}(0)}{1!} R + \frac{\Lambda_2^{''}(0)}{2!} R^2 + \ldots\right] \qquad (E.8)$$

For Λ_1 and Λ_2 to be expanded about R=0. These Λ and their derivatives can then be found and the ζ and μ equations rewritten as

$$\zeta = \left(\frac{R}{2}\right)^{\frac{1}{2}} \frac{1}{d} \left\{(2d)^{\frac{1}{2}} + \frac{1}{2(2d)^{\frac{1}{2}}} R + \frac{(1-\sin\alpha)}{4(2d)^{3/2}} \frac{R^2}{2} + \,.. \right\}(1+\sin\alpha)^{\frac{1}{2}} \qquad (E.9)$$

$$\mu = (2R)^{\frac{1}{2}} \cos\alpha \left\{(2d)^{-\frac{1}{2}} - \frac{1}{2(2d)^{3/2}} R + \frac{(1+2\sin\alpha)}{4(2d)^{5/2}} \frac{R^2}{2} + \,.. \right\} (1+\sin\alpha)^{-\frac{1}{2}} \qquad (E.10)$$

With the trigonometric identities

$$(1+\sin\alpha)^{\frac{1}{2}} = \sin\alpha/2 + \cos\alpha/2$$

and

$$\cos\alpha(1+\sin\alpha)^{-\frac{1}{2}} = \cos\alpha/2 - \sin\alpha/2$$

Equations E.9 and E.10 can be recast as

$$\zeta = \left(\frac{R}{d}\right)^{\frac{1}{2}} \left\{1+\frac{1}{4}\left(\frac{R}{d}\right)+\frac{1}{32}\left(\frac{R}{d}\right)^2(1-2\sin\alpha) + \ldots\right\}(\cos\alpha/2 + \sin\alpha/2) \qquad (E.11)$$

$$\mu = \left(\frac{R}{d}\right)^{\frac{1}{2}} \left\{1-\frac{1}{4}\left(\frac{R}{d}\right)+\frac{1}{32}\left(\frac{R}{d}\right)^2(1+2\sin\alpha) + \ldots\right\}(\cos\alpha/2 - \sin\alpha/2 \qquad (E.12)$$

These are Equations 3.50 and 3.51 given in the text.

APPENDIX F

DERIVATION OF PRESSURE FIELD EQUATION

The integrals in Equation 3.67 are carried out in this appendix. Equation 3.67 can be written for F(x)=0 as

$$p(x,\mu)(d\zeta_o)^3/(\rho\nu^2 d)= \sum_{n=1} G'_{2n-1}(x) \int_o^{\mu} \frac{P'_{2n-1}(\mu)}{(x^2+\mu^2\zeta_o^{-2})}\, d\mu-2xG_{2n-1}(x)$$

$$\int_o^{\mu} \frac{P'_{2n-1}(\mu)}{(x^2+\mu^2\zeta_o^{-2})^2}\, d\mu \qquad (F.1)$$

where the series terminate at N=3. G_{2n-1} and P'_{2n-1} equations are given as follows:

$$G_{2n-1}(x)=(x^2+\zeta_o^{-2})\, g''_{2n-1}(x)-2n(2n-1)g_{2n-1}(x)$$

$$G'_{2n-1}(x)=(x^2+\zeta_o^{-2})g'''_{2n-1}(x)+2xg''_{2n-1}(x)-2n(2n-1)g'_{2n-1}(x)$$

$$P'_1(\mu)=1$$

$$P'_3(\mu)=3/2(5\mu^2-1)$$

$$P'_5(\mu)=15/8(21\mu^4-14\mu^2+1)$$

Integrals in Equations F.1 can be carried out with the help of the formulae given in [41]. There is derived that

$$\begin{aligned}
& p(x;\mu)(d\zeta_o)^3/(\rho\nu^2 d)=[-\frac{G_1}{x^2}+(\frac{1}{x^2}-5\zeta_o^2)3/2G_3+(63x^2\zeta_o^4) \\
& +14\zeta_o^2-1/x^2)15/8G_5+G_1'/x-(\frac{1}{x}+5x\zeta_o^2)3/2\ G_3'+(21x^3\zeta_o^4 \\
& +14\zeta_o^2x+1/x)15/8\ G_5']\zeta_o\tan^{-1}(\mu/x\zeta_o)+[-G_1/x+(\frac{1}{x}+5x\zeta_o^2)3/2\ G_3 \\
& +(21x\mu^2\zeta^2{}_o-14\zeta_o^2x-1/x)15/8\ G_5]\ \mu/(x^2+\mu^2\zeta_o^{-2}) \\
& +\{G_3'+[21(\frac{\mu^2}{3}\ x^2\zeta_o^2)-14]G_5'/4-63/4\ x\zeta_o^2G_5\}\ 15/2\mu\zeta_o^2 \qquad (F.2)
\end{aligned}$$

APPENDIX G

DERIVATION OF EQUATION 4.2

Both sides of the energy equation

$$(V \cdot \nabla)T = \alpha\nabla^2 T \tag{G.1}$$

can be expressed in terms of the general curvilinear coordinates (ξ_1, ξ_2, ξ_3) as

$$(V \cdot \nabla)T = \frac{V_1}{h_1}\frac{\partial T}{\partial \xi_1} + \frac{V_2}{h_2}\frac{\partial}{\partial \xi_2} \tag{G.2}$$

and

$$\nabla^2 T = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \xi_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial T}{\xi_1}\right) + \frac{\partial}{\partial \xi_2}\left(\frac{h_1 h_3}{h_2}\frac{T}{\partial \xi_2}\right)\right] \tag{G.3}$$

where V_1 and V_2 are velocity components of V, and h_1, h_2 and h_3 are those scale factors defined in Appendix A.

Upon substituting of Equations G.2 and G.3 and making use of those h equations mentioned above changes Equation G.1 to

$$(\xi_1^2+d^2)^{\frac{1}{2}}V_1\frac{\partial T}{\partial \xi_1} + (1-\xi_2^2)^{\frac{1}{2}}V_2\frac{\partial T}{\partial \xi_2} = \frac{\alpha}{(\xi_1^2+d^2\xi_2^2)^{\frac{1}{2}}}\left\{\frac{\partial}{\partial \xi_1}\left[(\xi_1^2+d^2)\frac{\partial T}{\partial \xi_1}\right] + \frac{\partial}{\partial \xi_2}\left[(1-\xi_2^2)\frac{\partial T}{\partial \xi_2}\right]\right\} \tag{G.4}$$

Notice that the curvilinear coordinates (ξ_1, ξ_2) are related to (x,μ) by

$$\xi_1=d\zeta x, \quad \xi_2=\mu$$

and for the velocity components,

$$V_1=u \quad \text{and} \quad V_2=v$$

Equation G.4 can then be expressed in terms of these new variables as

$$(x^2+\zeta_o^{-2})^{\frac{1}{2}} u\frac{\partial T}{\partial x} + (1-\mu^2)^{\frac{1}{2}} v \frac{\partial T}{\partial \mu} = \frac{\alpha}{\partial\zeta_o(x^2+\mu^2\zeta_o^{-2})^{\frac{1}{2}}} \{\frac{\partial}{\partial x} [(x^2+\zeta_o^{-2}) \frac{\partial T}{\partial x}]$$

$$+ \frac{\partial}{\partial \mu} (1-\mu^2) \frac{\partial T}{\partial \mu}]\} \qquad (G.5)$$

Upon using the relationships between the velocity components and the stream function (Equations A.5 and A.6), Equation G.5 can be changed to

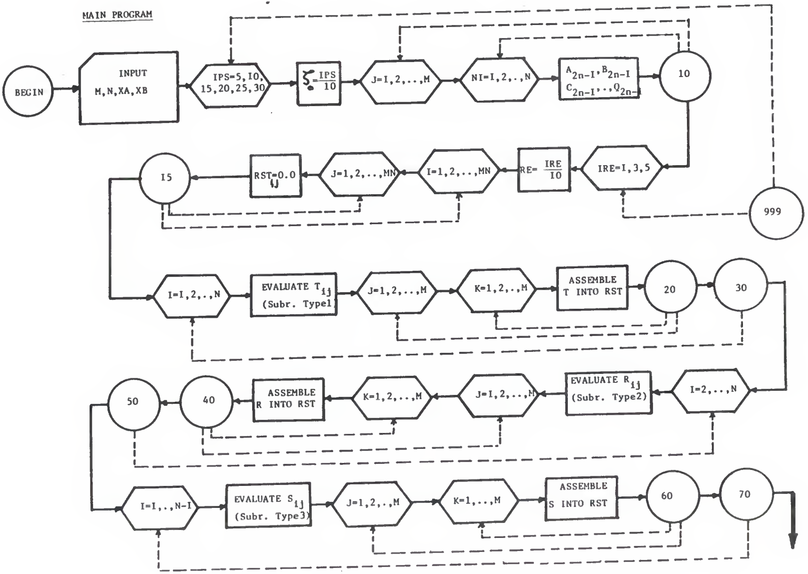
$$\frac{\partial T}{\partial \mu}\frac{\partial \Psi}{\partial x} - \frac{\partial}{\partial x}\frac{\partial \Psi}{\partial \mu} = \alpha d\zeta_o \cdot \{\frac{\partial}{\partial x} [(x^2+\zeta_o^{-2}) \frac{\partial T}{\partial x}] + \frac{\partial}{\partial \mu} [(1-\mu^2) \frac{\partial}{\partial \mu}]\} \qquad (G.6)$$

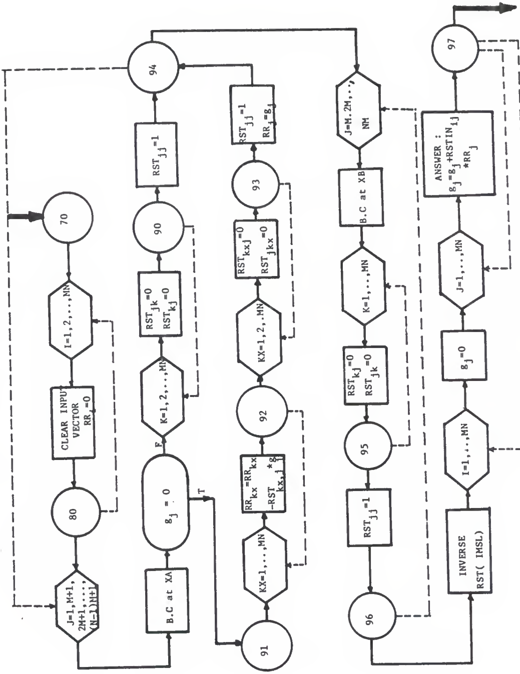
which is Equation 4.2.

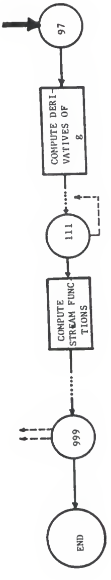
APPENDIX H

COMPUTER PROGRAM AND SAMPLE DATA FOR STREAM FUNCTIONS, VELOCITY AND PRESSURE FIELDS

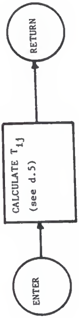
MAIN PROGRAM



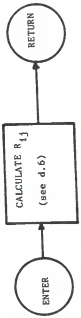

70
I=1,2,...,MN
CLEAR INPUT VECTOR RRi=0
80
J=1,M+1, 2M+1,...., (N-1)M+1
B.C at XA
gj = 0
T
F
K=1,2,...,MN
RSTjk=0 RSTkj=0
90
RSTjj=1
94
91
KX=1,...,MN
RRkx=RRkx -RSTkx,j *gj
92
KX=1,2,..MN
RSTkxj=0 RSTjkx=0
93
RSTjj=1 RRj=gj
J=M,2M,...,NM
B.C at XB
K=1,...,MN
RSTkj=0 RSTjk=0
95
RSTjj=1
96
INVERSE RST(IMSL)
I=1,...,MN
gj=0
J=1,...,MN
ANSWER : gj=gj+RSTINij *RRj
97



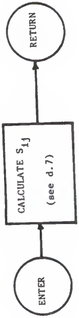
SUBROUTINE TYPE1(L,M)



SUBROUTINE TYPE2(L,M)



SUBROUTINE TYPE3(L,M)



```
C                        *** MAIN PROGRAM ***
C
C
C     ******************************************************************
C     **   SOLUTION OF SET OF N 4TH ORDER ORDINARY DIFF. EQUATIONS    **
C     ** TO DETERMINE G(X) FROM WHICH THE STREAM FUNCTION , VELOCITY  **
C     **                   AND PRESSURE ARE DEFINED                   **
C     **            (THIS PROGRAM WAS WRITTEN BY L. AMMAR)            **
C     ******************************************************************
C
C        *  M = NUMBER OF POINTS IN THE X-DIRECTION
C        *  N = NUMBER OF SET OF 4TH ORDER EQUATIONS ( INPUT )
C        *  MN = DIMENSION OF GLOBAL MATRIX = M*N
C        *  T = COEFFICIENTS OF MATRIX OF TYPE 1
C        *  R = COEFFICIENTS OF MATRIX OF TYPE 2
C        *  S = COEFFICIENTS OF MATRIX OF TYPE 3
C        *  RST = COEFFICIENTS OF GLOBAL MATRIX
C        *  RR = INPUT VECTOR
C        *  DX = STEP SIZE
C        *  XA,XB = END POINTS
C        *  X,MU = OBLATE SPHEROIDAL COORDINATES
C        *  ZETA = FIXES THE SIZE OF SPHEROIDAL CAP / FEEDER TUBE
C        *  RE = REYNOLDS NUMBER
C        *  A,B,C,...,Q = COEFFICIENTS OF DIFF. EQUATIONS
C        *  WKAREA = WORK AREA OF IMSL OF DIMENSION = MN**2+3.*MN
C        *  PSI = STREAM FUNCTION
C        *  W = DIMENSIONLESS VELOCITY (V*C**2/(NU*D))
C        *  P = DIMENSIONLESS PRESSURE (DEFINED IN THE TEXT)
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION RST(303,303),RSTINV(303,303),WKAREA(92718),RR(303),
     A          G(303),GP(303),GPP(303),GPPP(303)
      COMMON/BLOK0/DX
      COMMON/BLOK1/A(101,3),B(101,3),C(101,3),D(101,3),E(101,3)
      COMMON/BLOK2/H(101,3),AI(101,3),AJ(101,3),AK(101,3)
      COMMON/BLOK3/AL(101,3),AM(101,3),AN(101,3),Q(101,3)
      COMMON/TYPET/T(101,101)
      COMMON/TYPER/R(101,101)
      COMMON/TYPES/S(101,101)
      M=101
      N=3
      MN=M*N
      XA=0.D0
      XB=1.D0
      IRST=MN
      IDGT=6
      M1=M-1
      DX=(XB-XA)/M1
```

```
C     ...INTRODUCE THE COEF. OF DIFF. EQUATIONS...
      DO 999 IPS=10,30,5
      DPS=IPS
      ZETA=DPS/10.
      DO 10 J=1,M
        DJ=J-1
        X=DJ*DX
        Z=X*X+1./(ZETA*ZETA)
      DO 10 NI=1,N
        DN=NI
       A(J,NI)=Z*Z*(X*X+(8.*DN*DN-4.*DN-3.)/((4.*DN+1.)*(4.*DN-3.)
     B         *ZETA*ZETA))
       B(J,NI)=-16.*X*Z*DN*(2.*DN-1.)/((4.*DN+1.)*(4.*DN-3.)*ZETA*ZETA)
       C(J,NI)=-4.*DN*(2.*DN-1.)*Z*(X*X+(8.*DN*DN-4.*DN-9.)
     C         /((4.*DN+1.)*(4.*DN-3.)*ZETA*ZETA))
       D(J,NI)=8.*X*Z*DN*(2.*DN-1.)
       E(J,NI)=4.*DN*(2.*DN-1.)*(DN+1.)*(2.*DN-3.)
     D         *(X*X+(8.*DN*DN-4.*DN+1.)/((4.*DN+1.)*(4.*DN-3.)
     E         *ZETA*ZETA))
       H(J,NI)=Z*Z*2.*(DN-1.)*(2.*DN-3.)/((4.*DN-5.)*(4.*DN-3.)
     F         *ZETA*ZETA)
       AI(J,NI)=8.*X*Z*(DN-1.)*(2.*DN-3.)/((4.*DN-5.)*(4.*DN-3.)
     G          *ZETA*ZETA)
       AJ(J,NI)=-8.*Z*(DN-1.)*(DN-2.)*(2.*DN-3.)*(2.*DN-3.)
     H          /((4.*DN-5.)*(4.*DN-3.)*ZETA*ZETA)
       AK(J,NI)=8.*(DN-1.)*(DN-1.)*(2.*DN-3.)*(2.*DN-3.)*(DN-2.)
     I            *(2.*DN-5.)/((4.*DN-5.)*(4.*DN-3.)*ZETA*ZETA)
       AL(J,NI)=2.*Z*Z*(DN+1.)*(2.*DN+1.)/((4.*DN+1.)*(4.*DN+3.)
     J          *ZETA*ZETA)
       AM(J,NI)=8.*X*Z*(DN+1.)*(2.*DN+1.)/((4.*DN+1.)*(4.*DN+3.)
     K          *ZETA*ZETA)
       AN(J,NI)=-8.*Z*(DN+1.)*(DN+1.)*(2.*DN+1.)*(2.*DN+3.)
     L               /((4.*DN+1.)*(4.*DN+3.)*ZETA*ZETA)
       Q(J,NI)=8.*(DN+1.)*(DN+1.)*(DN+2.)*(2.*DN+1.)*(2.*DN+1.)
     M           *(2.*DN+3.)/((4.*DN+1.)*(4.*DN+3.)*ZETA*ZETA)
 10   CONTINUE
C
      DO 999 IRE=1,3
      DRE=IRE
      RE=DRE/10.
C     ...CLEAR GLOBAL MATRIX RST...

      DO 15 I=1,MN
      DO 15 J=1,MN
 15     RST(I,J)=0.D0
C     ...COMPUTE COEFFICIENTS OF MATRIX T...
      DO 30 I=1,N
      CALL TYPE1(I,M)
C
C     ...ASSEMBLE T INTO GLOBAL MATRIX RST...
      DO 20 J=1,M
      DO 20 K=1,M
      JI=M*(I-1)+J
      KI=M*(I-1)+K                                                      SAMI102
         RST(JI,KI)=T(J,K)                                              SAMI103
   20 CONTINUE                                                          SAMI104
   30 CONTINUE                                                          SAMI105
C                                                                       SAMI106
C     ...COMPUTE COEFFICIENTS OF MATRIX R...                            SAMI107
      DO 50 I=2,N                                                       SAMI108
      CALL TYPE2(I,M)                                                   SAMI109
C                                                                       SAMI110
C     ...ASSEMBLE R INTO GLOBAL MATRIX RST...                           SAMI111
      DO 40 J=1,M                                                       SAMI112
      DO 40 K=1,M                                                       SAMI113
      JI=M*(I-1)+J                                                      SAMI114
      KI=M*(I-2)+K                                                      SAMI115
         RST(JI,KI)=R(J,K)                                              SAMI116
   40 CONTINUE                                                          SAMI117
   50 CONTINUE                                                          SAMI118
C                                                                       SAMI119
C     ...COMPUTE COEFFICIENTS OF MATRIX S...                            SAMI120
      N1=N-1                                                            SAMI121
      DO 70 I=1,N1                                                      SAMI122
      CALL TYPE3(I,M)                                                   SAMI123
C                                                                       SAMI124
C     ...ASSEMBLE S INTO GLOBAL MATRIX RST...                           SAMI125
      DO 60 J=1,M                                                       SAMI126
      DO 60 K=1,M                                                       SAMI127
      JI=M*(I-1)+J                                                      SAMI128
      KI=M*I+K                                                          SAMI129
         RST(JI,KI)=S(J,K)                                              SAMI130
   60 CONTINUE                                                          SAMI131
   70 CONTINUE                                                          SAMI132
C                                                                       SAMI133
C     ...CLEAR INPUT VECTOR RR...                                       SAMI134
      DO 80 I=1,MN                                                      SAMI135
   80    RR(I)=0.D0                                                     SAMI136
C     ...INTRODUCE DIRICHLET BC AT XA...                                SAMI137
      MMM=(N-1)*M+1                                                     SAMI138
      DO 94 J=1,MMM,M                                                   SAMI139
         G(J)=0.D0                                                      SAMI140
         G(1)=3./10.*RE                                                 SAMI141
         G(M+1)=1./30.*RE                                               SAMI142
      IF(G(J).NE.0.) GO TO 91                                           SAMI143
      DO 90 K=1,MN                                                      SAMI144
         RST(J,K)=0.D0                                                  SAMI145
   90    RST(K,J)=0.D0                                                  SAMI146
         RST(J,J)=1.D0                                                  SAMI147
      GO TO 94                                                          SAMI148
C                                                                       SAMI149
C     ...COMPUTE INPUT VECTOR RR(I) MODIFIED BY BC AT XA...             SAMI150
   91 CONTINUE                                                          SAMI151
      DO 92 KX=1,MN                                                     SAMI152
   92    RR(KX)=RR(KX)-RST(KX,J)*G(J)                                   SAMI153
```

```
      DO 93 KX=1,MN
         RST(KX,J)=0.D0
         RST(J,KX)=0.D0
 93   CONTINUE
         RST(J,J)=1.D0
         RR(J)=G(J)
 94   CONTINUE
C
C     ...INTRODUCE DIRICHLET BC AT XB...
      DO 96 J=M,MN,M
         G(J)=0.D0
      DO 95 K=1,MN
         RST(J,K)=0.D0
 95      RST(K,J)=0.D0
         RST(J,J)=1.D0
 96   CONTINUE
C
C     ...INVERT GLOBAL MATRIX RST AND MULTIPLY BY VECTOR RR...
C     ...USING IMSL LIBRARY...
      CALL LINV2F(RST,MN,IRST,RSTINV,IDGT,WKAREA,IER)
      DO 97 I=1,MN
         G(I)=0.D0
      DO 97 J=1,MN
 97      G(I)=G(I)+RSTINV(I,J)*RR(J)
C
C     ...COMPUTE DERIVATIVES OF G...
      DO 98 J=1,MMM,M
         GP(J)=0.D0
         GPP(J)=2./(DX*DX)*(G(J+1)-G(J))
         GPPP(J)=1./(2.*DX*DX*DX)*(-5.*G(J)+18.*G(J+1)-24.*G(J+2)
     N            +14.*G(J+3)-3.*G(J+4))
 98   CONTINUE
      DO 99 J=M,MN,M
         GP(J)=0.D0
         GPP(J)=2./(DX*DX)*(G(J-1)-G(J))
         GPPP(J)=1./(2.*DX*DX*DX)*(5.*G(J)-18.*G(J-1)+24.*G(J-2)
     Q            -14.*G(J-3)+3.*G(J-4))
 99   CONTINUE
      DO 101 J=2,M1
      MJ=(N-1)*M+J
      DO 101 JX=J,MJ,M
         GP(JX)=1./(2.*DX)*(G(JX+1)-G(JX-1))
 101     GPP(JX)=1./(DX*DX)*(G(JX-1)-2.*G(JX)+G(JX+1))
      LJ=(N-1)*M+2
      DO 102 JX=2,LJ,M
         GPPP(JX)=1./(2.*DX*DX*DX)*(-G(JX)+2.*G(JX-1)-2.*G(JX+1)
     R            +G(JX+2))
 102  CONTINUE
      MN1=MN-1
      DO 103 JX=M1,MN1,M
         GPPP(JX)=1./(2.*DX*DX*DX)*(G(JX)-2.*G(JX+1)+2.*G(JX-1)-G(JX-2))
 103  CONTINUE
```

```
      M2=M-2                                                            SAMI206
      DO 104 J=3,M2                                                     SAMI207
      MJ=(N-1)*M+J                                                      SAMI208
      DO 104 JX=J,MJ,M                                                  SAMI209
         GPPP(JX)=1./(2.*DX*DX*DX)*(-G(JX-2)+2.*G(JX-1)-2.*G(JX+1)      SAMI210
     S           +G(JX+2))                                              SAMI211
  104 CONTINUE                                                          SAMI212
      WRITE(6,222)N,ZETA,RE                                             SAMI213
      DO 111 IPR=1,M                                                    SAMI214
      DJ=IPR-1                                                          SAMI215
      X=DJ*DX                                                           SAMI216
      WRITE(6,333)X                                                     SAMI217
      IIP=(N-1)*M+IPR                                                   SAMI218
      KPR=-1                                                            SAMI219
      DO 111 JPR=IPR,IIP,M                                              SAMI220
      KPR=KPR+2                                                         SAMI221
      WRITE(6,444)KPR,G(JPR),KPR,GP(JPR),KPR,GPP(JPR),KPR,GPPP(JPR)     SAMI222
  111 CONTINUE                                                          SAMI223
      WRITE(6,555)IER                                                   SAMI224
C                                                                       SAMI225
C     ...COMPUTATION OF THE STREAM FUNCTION (DIMENSIONLESS)...          SAMI226
      WRITE(6,888)                                                      SAMI227
      DO 105 J=1,M                                                      SAMI228
      DJ=J-1                                                            SAMI229
      X=DJ*DX                                                           SAMI230
      WRITE(6,666)X                                                     SAMI231
      DO 105 K=1,11                                                     SAMI232
      DK=K-1                                                            SAMI233
      U=DK/10.                                                          SAMI234
      UU=U*U                                                            SAMI235
      UUUU=UU*UU                                                        SAMI236
      PSI=(1.-UU)*(G(J)+G(M+J)*(5.*UU-1.)*3./2.+G(2*M+J)                SAMI237
     T           *(21.*UUUU-14.*UU+1.)*15./8.)                          SAMI238
  105 WRITE(6,777)U,PSI                                                 SAMI239
C                                                                       SAMI240
C     ...COMPUTATION OF SPEED DISTRIBUTION...                           SAMI241
      WRITE(6,889)                                                      SAMI242
      DO 107 J=1,M                                                      SAMI243
      DJ=J-1                                                            SAMI244
      X=DJ*DX                                                           SAMI245
      WRITE(6,666)X                                                     SAMI246
      DO 107 K=1,11                                                     SAMI247
      DK=K-1                                                            SAMI248
      U=DK/10.                                                          SAMI249
      UU=U*U                                                            SAMI250
      UUU=UU*U                                                          SAMI251
      UUUU=UUU*U                                                        SAMI252
      UUUUU=UUUU*U                                                      SAMI253
      Z=X*X+UU/(ZETA*ZETA)                                              SAMI254
      ZZ=X*X+1./(ZETA*ZETA)                                             SAMI255
      IF(X.NE.0.OR.U.NE.0.) GO TO 106                                   SAMI256
      W=0.D0                                                            SAMI257
      GO TO 107                                                         SAMI258
```

```
 106  W=1./DSQRT(Z)*DSQRT(4./ZZ*(G(J)*U+3.*G(M+J)*(5.*UUU-3.*U)          SAMI259
     U  +15./8.*G(2*M+J)*(63.*UUUUU-70.*UUU+15.*U))**2                  SAMI260
     V  +(1.-UU)*(GP(J)+3./2.*GP(M+J)*(5.*UU-1.)                        SAMI261
     W  +15./8.*GP(2*M+J)*(21.*UUUU-14.*UU+1.))**2)                     SAMI262
 107  WRITE(6,890)U,W                                                   SAMI263
C                                                                       SAMI264
C     ...COMPUTATION OF THE PRESSURE...                                 SAMI265
      WRITE(6,891)                                                      SAMI266
      DO 108 J=2,M                                                      SAMI267
      DJ=J-1                                                            SAMI268
      X=DJ*DX                                                           SAMI269
      WRITE(6,666)X                                                     SAMI270
      ZZ=X*X+1./(ZETA*ZETA)                                             SAMI271
      G1=ZZ*GPP(J)-2.*G(J)                                              SAMI272
      GP1=ZZ*GPPP(J)+2.*X*GPP(J)-2.*GP(J)                               SAMI273
      G3=ZZ*GPP(M+J)-12.*G(M+J)                                         SAMI274
      GP3=ZZ*GPPP(M+J)+2.*X*GPP(M+J)-12.*GP(M+J)                        SAMI275
      G5=ZZ*GPP(2*M+J)-30.*G(2*M+J)                                     SAMI276
      GP5=ZZ*GPPP(2*M+J)+2.*X*GPP(2*M+J)-30.*GP(2*M+J)                  SAMI277
      Z1=1./(X*X)-5.*ZETA*ZETA                                          SAMI278
      Z2=63.*X*X*ZETA*ZETA*ZETA*ZETA-14.*ZETA*ZETA-1./(X*X)             SAMI279
      Z3=1./X+5.*X*ZETA*ZETA                                            SAMI280
      Z4=21.*X*X*X*ZETA*ZETA*ZETA*ZETA+14.*ZETA*ZETA*X+1./X             SAMI281
      DO 108 K=1,11                                                     SAMI282
      DK=K-1                                                            SAMI283
      U=DK/10.                                                          SAMI284
      UU=U*U                                                            SAMI285
      Z=X*X+UU/(ZETA*ZETA)                                              SAMI286
      P=(-G1/(X*X)+3./2.*Z1*G3+15./8.*Z2*G5+GP1/X-3./2.*Z3*GP3          SAMI287
     X  +15./8.*Z4*GP5)*ZETA*DATAN(U/(X*ZETA))                          SAMI288
     Y  +(-G1/X+3./2.*Z3*G3+15./8.*(21.*X*UU*ZETA*ZETA_14.*X*ZETA*ZETA  SAMI289
     Z  -1./X)*G5)*U/Z+(GP3+(21.*(UU/3.-X*X*ZETA*ZETA)-14.)*GP5/4.      SAMI290
     A  -63./4.*X*ZETA*ZETA*G5)*15./2.*U*ZETA*ZETA                      SAMI291
 108  WRITE(6,892)U,P                                                   SAMI292
 999  CONTINUE                                                          SAMI293
         STOP                                                           SAMI294
 222  FORMAT(1H1,5X,'N=',I2,5X,'ZETA=',F5.3,5X,'RE=',F5.3,///)          SAMI295
 333  FORMAT(3X,'X=',F4.2)                                              SAMI296
 444  FORMAT(14X,'G',I2,'=',E11.4,5X,'GP',I2,'=',E11.4,5X,'GPP',I2,'=',ESAMI297
     U11.4,5X,'GPPP',I2,'=',E11.4,/)                                    SAMI298
 555  FORMAT(5X,'IER=',I3,////)                                         SAMI299
 666  FORMAT(3X,'X=',F4.2)                                              SAMI300
 777  FORMAT(20X,'MU=',F4.2,5X,'PSI=',E15.5,/)                          SAMI301
 888  FORMAT(30X,'DIMENSIONLESS  STREAM FUNCTION  PSI(X,MU) ',///)      SAMI302
 889  FORMAT(1H1,30X,'SPEED DISTRIBUTION V*C**2/(NU*D)',///)            SAMI303
 890  FORMAT(20X,'MU=',F4.2,5X,'W=',E15.5,/)                            SAMI304
 891  FORMAT(1H1,50X,'PRESSURE DISTRIBUTION',///)                       SAMI305
 892  FORMAT(20X,'MU=',F4.2,'P=',E15.5,/)                               SAMI306
         END                                                            SAMI307
```

```
C                                                                       SAMI308
C                             *** TYPE1 ***                             SAMI309
C                                                                       SAMI310
      SUBROUTINE TYPE1(L,M)                                             SAMI311
      IMPLICIT REAL*8(A-H,O-Z)                                          SAMI312
      COMMON/BLOK0/DX                                                   SAMI313
      COMMON/BLOK1/A(101,3),B(101,3),C(101,3),D(101,3),E(101,3)         SAMI314
      COMMON/TYPET/T(101,101)                                           SAMI315
C     ...CLEAR MATRIX T...                                              SAMI316
      DO 100 I=1,M                                                      SAMI317
      DO 100 J=1,M                                                      SAMI318
  100    T(I,J)=0.D0                                                    SAMI319
C     ...COMPUTE T FOR NI=L...                                          SAMI320
         T(2,2)=7.*A(2,L)-B(2,L)*DX/2.-2.*C(2,L)*DX*DX                  SAMI321
     1         +E(2,L)*DX*DX*DX*DX                                      SAMI322
         T(M-1,M-1)=7.*A(M-1,L)+B(M-1,L)*DX/2.-2.*C(M-1,L)*DX*DX        SAMI323
     2               +E(M-1,L)*DX*DX*DX*DX                              SAMI324
      M2=M-2                                                            SAMI325
      DO 110 I=3,M2                                                     SAMI326
  110    T(I,I)=6.*A(I,L)-2.*C(I,L)*DX*DX+E(I,L)*DX*DX*DX*DX            SAMI327
      M1=M-1                                                            SAMI328
      DO 120 I=2,M1                                                     SAMI329
         T(I,I-1)=-4.*A(I,L)+B(I,L)*DX+C(I,L)*DX*DX-D(I,L)*DX*DX*DX/2.  SAMI330
  120    T(I,I+1)=-4.*A(I,L)-B(I,L)*DX+C(I,L)*DX*DX+D(I,L)*DX*DX*DX/2.  SAMI331
      DO 130 I=2,M2                                                     SAMI332
         T(I+1,I-1)=A(I+1,L)-B(I+1,L)*DX/2.                             SAMI333
  130    T(I,I+2)=A(I,L)+B(I,L)*DX/2.                                   SAMI334
      RETURN                                                            SAMI335
      END                                                               SAMI336
```

```
C                                                                       SAMI337
C                             *** TYPE2 ***                             SAMI338
C                                                                       SAMI339
      SUBROUTINE TYPE2(L,M)                                             SAMI340
      IMPLICIT REAL*8(A-H,O-Z)                                          SAMI341
      COMMON/BLOKO/DX                                                   SAMI342
      COMMON/BLOK2/H(101,3),AI(101,3),AJ(101,3),AK(101,3)               SAMI343
      COMMON/TYPER/R(101,101)                                           SAMI344
C     ...CLEAR MATRIX R...                                              SAMI345
      DO 200 I=1,M                                                      SAMI346
      DO 200 J=1,M                                                      SAMI347
 200     R(I,J)=0.D0                                                    SAMI348
C     ...COMPUTE R FOR NI=L...                                          SAMI349
         R(2,2)=7.*H(2,L)-AI(2,L)*DX/2.-2.*AJ(2,L)*DX*DX                SAMI350
     3          +AK(2,L)*DX*DX*DX*DX                                    SAMI351
         R(M-1,M-1)=7.*H(M-1,L)+AI(M-1,L)*DX/2.-2.*AJ(M-1,L)*DX*DX      SAMI352
     4              +AK(M-1,L)*DX*DX*DX*DX                              SAMI353
      M2=M-2                                                            SAMI354
      DO 210 I=3,M2                                                     SAMI355
 210     R(I,I)=6.*H(I,L)-2.*AJ(I,L)*DX*DX+AK(I,L)*DX*DX*DX*DX          SAMI356
      M1=M-1                                                            SAMI357
      DO 220 I=2,M1                                                     SAMI358
         R(I,I-1)=-4.*H(I,L)+AI(I,L)*DX+AJ(I,L)*DX*DX                   SAMI359
 220     R(I,I+1)=-4.*H(I,L)-AI(I,L)*DX+AJ(I,L)*DX*DX                   SAMI360
      DO 230 I=2,M2                                                     SAMI361
         R(I+1,I-1)=H(I+1,L)-AI(I+1,L)*DX/2.                            SAMI362
 230     R(I,I+2)=H(I,L)+AI(I,L)*DX/2.                                  SAMI363
      RETURN                                                            SAMI364
      END                                                               SAMI365
```

```
C
C                            *** TYPE3 ***
C
      SUBROUTINE TYPE3(L,M)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/BLOK0/DX
      COMMON/BLOK3/AL(101,3),AM(101,3),AN(101,3),Q(101,3)
      COMMON/TYPES/S(101,101)
C     ...CLEAR MATRIX S...
      DO 300 I=1,M
      DO 300 J=1,M
 300     S(I,J)=0.D0
C     ...COMPUTE S FOR NI=L...
         S(2,2)=7.*AL(2,L)-AM(2,L)*DX/2.-2.*AN(2,L)*DX*DX
     5          +Q(2,L)*DX*DX*DX*DX
         S(M-1,M-1)=7.*AL(M-1,L)+AM(M-1,L)*DX/2.-2.*AN(M-1,L)*DX*DX
     6              +Q(M-1,L)*DX*DX*DX*DX
      M2=M-2
      DO 310 I=3,M2
 310     S(I,I)=6.*AL(I,L)-2.*AN(I,L)*DX*DX+Q(I,L)*DX*DX*DX*DX
      M1=M-1
      DO 320 I=2,M1
         S(I,I-1)=-4.*AL(I,L)+AM(I,L)*DX+AN(I,L)*DX*DX
 320     S(I,I+1)=-4.*AL(I,L)-AM(I,L)*DX+AN(I,L)*DX*DX
      DO 330 I=2,M2
         S(I+1,I-1)=AL(I+1,L)-AM(I+1,L)*DX/2.
 330     S(I,I+2)=AL(I,L)+AM(I,L)*DX/2.
      RETURN
      END
```

A Sample of the Computer Output

```
 N= 3      ZETA=2.000      RE=0.100
X=0.0
          G 1= 0.3000D-01     GP 1= 0.0            GPP 1=-0.9699D-01    GPPP 1= 0.1082D+01
          G 3= 0.3333D-02     GP 3= 0.0            GPP 3=-0.6475D-01    GPPP 3= 0.6196D+00
          G 5= 0.0            GP 5= 0.0            GPP 5=-0.1542D-04    GPPP 5= 0.2698D-03
X=0.01
          G 1= 0.3000D-01     GP 1=-0.9188D-03     GPP 1=-0.8678D-01    GPPP 1= 0.9456D+00
          G 3= 0.3330D-02     GP 3=-0.6174D-03     GPP 3=-0.5873D-01    GPPP 3= 0.5808D+00
          G 5=-0.7708D-09     GP 5=-0.1413D-06     GPP 5=-0.1284D-04    GPPP 5= 0.2328D-03
X=0.02
          G 1= 0.2998D-01     GP 1=-0.1743D-02     GPP 1=-0.7808D-01    GPPP 1= 0.7989D+00
          G 3= 0.3321D-02     GP 3=-0.1177D-02     GPP 3=-0.5313D-01    GPPP 3= 0.5399D+00
          G 5=-0.2826D-08     GP 5=-0.2593D-06     GPP 5=-0.1076D-04    GPPP 5= 0.1874D-03
X=0.03
          G 1= 0.2996D-01     GP 1=-0.2488D-02     GPP 1=-0.7080D-01    GPPP 1= 0.6623D+00
          G 3= 0.3307D-02     GP 3=-0.1682D-02     GPP 3=-0.4793D-01    GPPP 3= 0.5011D+00
          G 5=-0.5956D-08     GP 5=-0.3586D-06     GPP 5=-0.9093D-05    GPPP 5= 0.1497D-03
X=0.04
          G 1= 0.2993D-01     GP 1=-0.3166D-02     GPP 1=-0.6483D-01    GPPP 1= 0.5358D+00
          G 3= 0.3287D-02     GP 3=-0.2137D-02     GPP 3=-0.4311D-01    GPPP 3= 0.4641D+00
          G 5=-0.9997D-08     GP 5=-0.4428D-06     GPP 5=-0.7766D-05    GPPP 5= 0.1186D-03
X=0.05
          G 1= 0.2990D-01     GP 1=-0.3790D-02     GPP 1=-0.6008D-01    GPPP 1= 0.4191D+00
          G 3= 0.3264D-02     GP 3=-0.2546D-02     GPP 3=-0.3865D-01    GPPP 3= 0.4292D+00
          G 5=-0.1481D-07     GP 5=-0.5153D-06     GPP 5=-0.6720D-05    GPPP 5= 0.9310D-04
X=0.06
          G 1= 0.2986D-01     GP 1=-0.4373D-02     GPP 1=-0.5645D-01    GPPP 1= 0.3122D+00
          G 3= 0.3236D-02     GP 3=-0.2912D-02     GPP 3=-0.3453D-01    GPPP 3= 0.3961D+00
          G 5=-0.2030D-07     GP 5=-0.5784D-06     GPP 5=-0.5904D-05    GPPP 5= 0.7225D-04
X=0.07
          G 1= 0.2981D-01     GP 1=-0.4924D-02     GPP 1=-0.5384D-01    GPPP 1= 0.2146D+00
          G 3= 0.3206D-02     GP 3=-0.3238D-02     GPP 3=-0.3073D-01    GPPP 3= 0.3649D+00
          G 5=-0.2638D-07     GP 5=-0.6343D-06     GPP 5=-0.5275D-05    GPPP 5= 0.5531D-04
X=0.08
          G 1= 0.2976D-01     GP 1=-0.5454D-02     GPP 1=-0.5216D-01    GPPP 1= 0.1263D+00
          G 3= 0.3172D-02     GP 3=-0.3528D-02     GPP 3=-0.2723D-01    GPPP 3= 0.3355D+00
          G 5=-0.3299D-07     GP 5=-0.6847D-06     GPP 5=-0.4798D-05    GPPP 5= 0.4161D-04
X=0.09
          G 1= 0.2970D-01     GP 1=-0.5972D-02     GPP 1=-0.5131D-01    GPPP 1= 0.4686D-01
          G 3= 0.3135D-02     GP 3=-0.3784D-02     GPP 3=-0.2402D-01    GPPP 3= 0.3079D+00
          G 5=-0.4007D-07     GP 5=-0.7309D-06     GPP 5=-0.4443D-05    GPPP 5= 0.3060D-04
X=0.10
          G 1= 0.2964D-01     GP 1=-0.6484D-02     GPP 1=-0.5122D-01    GPPP 1=-0.2401D-01
          G 3= 0.3096D-02     GP 3=-0.4010D-02     GPP 3=-0.2107D-01    GPPP 3= 0.2820D+00
          G 5=-0.4761D-07     GP 5=-0.7740D-06     GPP 5=-0.4186D-05    GPPP 5= 0.2179D-04
```

```
X=0.11
          G 1= 0.2957D-01     GP 1=-0.6999D-02     GPP 1=-0.5179D-01     GPPP 1=-0.8667D-01
          G 3= 0.3055D-02     GP 3=-0.4207D-02     GPP 3=-0.1838D-01     GPPP 3= 0.2578D+00
          G 5=-0.5555D-07     GP 5=-0.8150D-06     GPP 5=-0.4007D-05     GPPP 5= 0.1481D-04
X=0.12
          G 1= 0.2950D-01     GP 1=-0.7523D-02     GPP 1=-0.5295D-01     GPPP 1=-0.1415D+00
          G 3= 0.3012D-02     GP 3=-0.4378D-02     GPP 3=-0.1592D-01     GPPP 3= 0.2352D+00
          G 5=-0.6390D-07     GP 5=-0.8545D-06     GPP 5=-0.3890D-05     GPPP 5= 0.9313D-05
X=0.13
          G 1= 0.2942D-01     GP 1=-0.8061D-02     GPP 1=-0.5462D-01     GPPP 1=-0.1888D+00
          G 3= 0.2967D-02     GP 3=-0.4526D-02     GPP 3=-0.1367D-01     GPPP 3= 0.2141D+00
          G 5=-0.7264D-07     GP 5=-0.8930D-06     GPP 5=-0.3821D-05     GPPP 5= 0.5034D-05
X=0.14
          G 1= 0.2934D-01     GP 1=-0.8618D-02     GPP 1=-0.5673D-01     GPPP 1=-0.2290D+00
          G 3= 0.2921D-02     GP 3=-0.4653D-02     GPP 3=-0.1163D-01     GPPP 3= 0.1945D+00
          G 5=-0.8177D-07     GP 5=-0.9311D-06     GPP 5=-0.3789D-05     GPPP 5= 0.1756D-05
X=0.15
          G 1= 0.2925D-01     GP 1=-0.9197D-02     GPP 1=-0.5920D-01     GPPP 1=-0.2626D+00
          G 3= 0.2874D-02     GP 3=-0.4760D-02     GPP 3=-0.9783D-02     GPPP 3= 0.1764D+00
          G 5=-0.9127D-07     GP 5=-0.9689D-06     GPP 5=-0.3786D-05     GPPP 5=-0.6952D-06
X=0.16
          G 1= 0.2915D-01     GP 1=-0.9803D-02     GPP 1=-0.6198D-01     GPPP 1=-0.2899D+00
          G 3= 0.2826D-02     GP 3=-0.4849D-02     GPP 3=-0.8106D-02     GPPP 3= 0.1596D+00
          G 5=-0.1011D-06     GP 5=-0.1007D-05     GPP 5=-0.3803D-05     GPPP 5=-0.2462D-05
X=0.17
          G 1= 0.2905D-01     GP 1=-0.1044D-01     GPP 1=-0.6500D-01     GPPP 1=-0.3113D+00
          G 3= 0.2777D-02     GP 3=-0.4923D-02     GPP 3=-0.6591D-02     GPPP 3= 0.1441D+00
          G 5=-0.1114D-06     GP 5=-0.1045D-05     GPP 5=-0.3835D-05     GPPP 5=-0.3658D-05
X=0.18
          G 1= 0.2894D-01     GP 1=-0.1110D-01     GPP 1=-0.6821D-01     GPPP 1=-0.3273D+00
          G 3= 0.2728D-02     GP 3=-0.4982D-02     GPP 3=-0.5224D-02     GPPP 3= 0.1298D+00
          G 5=-0.1220D-06     GP 5=-0.1084D-05     GPP 5=-0.3876D-05     GPPP 5=-0.4370D-05
X=0.19
          G 1= 0.2883D-01     GP 1=-0.1180D-01     GPP 1=-0.7155D-01     GPPP 1=-0.3382D+00
          G 3= 0.2678D-02     GP 3=-0.5028D-02     GPP 3=-0.3995D-02     GPPP 3= 0.1166D+00
          G 5=-0.1331D-06     GP 5=-0.1123D-05     GPP 5=-0.3922D-05     GPPP 5=-0.4667D-05
X=0.20
          G 1= 0.2871D-01     GP 1=-0.1254D-01     GPP 1=-0.7497D-01     GPPP 1=-0.3444D+00
          G 3= 0.2627D-02     GP 3=-0.5062D-02     GPP 3=-0.2891D-02     GPPP 3= 0.1046D+00
          G 5=-0.1445D-06     GP 5=-0.1162D-05     GPP 5=-0.3970D-05     GPPP 5=-0.4604D-05
X=0.21
          G 1= 0.2858D-01     GP 1=-0.1330D-01     GPP 1=-0.7844D-01     GPPP 1=-0.3463D+00
          G 3= 0.2576D-02     GP 3=-0.5086D-02     GPP 3=-0.1903D-02     GPPP 3= 0.9353D-01
          G 5=-0.1563D-06     GP 5=-0.1202D-05     GPP 5=-0.4014D-05     GPPP 5=-0.4223D-05
X=0.22
          G 1= 0.2844D-01     GP 1=-0.1410D-01     GPP 1=-0.8190D-01     GPPP 1=-0.3443D+00
          G 3= 0.2525D-02     GP 3=-0.5101D-02     GPP 3=-0.1021D-02     GPPP 3= 0.8344D-01
          G 5=-0.1685D-06     GP 5=-0.1242D-05     GPP 5=-0.4054D-05     GPPP 5=-0.3556D-05
X=0.23
          G 1= 0.2830D-01     GP 1=-0.1494D-01     GPP 1=-0.8532D-01     GPPP 1=-0.3386D+00
          G 3= 0.2474D-02     GP 3=-0.5107D-02     GPP 3=-0.2346D-03     GPPP 3= 0.7423D-01
          G 5=-0.1812D-06     GP 5=-0.1283D-05     GPP 5=-0.4086D-05     GPPP 5=-0.2629D-05
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X=0.24
          G 1= 0.2814D-01     GP 1=-0.1581D-01     GPP 1=-0.8867D-01     GPPP 1=-0.3297D+00
          G 3= 0.2423D-02     GP 3=-0.5106D-02     GPP 3= 0.4638D-03     GPPP 3= 0.6586D-01
          G 5=-0.1942D-06     GP 5=-0.1324D-05     GPP 5=-0.4107D-05     GPPP 5=-0.1462D-05
X=0.25
          G 1= 0.2798D-01     GP 1=-0.1671D-01     GPP 1=-0.9192D-01     GPPP 1=-0.3179D+00
          G 3= 0.2372D-02     GP 3=-0.5098D-02     GPP 3= 0.1083D-02     GPPP 3= 0.5826D-01
          G 5=-0.2076D-06     GP 5=-0.1365D-05     GPP 5=-0.4115D-05     GPPP 5=-0.7167D-07
X=0.26
          G 1= 0.2781D-01     GP 1=-0.1765D-01     GPP 1=-0.9503D-01     GPPP 1=-0.3035D+00
          G 3= 0.2321D-02     GP 3=-0.5085D-02     GPP 3= 0.1629D-02     GPPP 3= 0.5139D-01
          G 5=-0.2215D-06     GP 5=-0.1406D-05     GPP 5=-0.4108D-05     GPPP 5= 0.1529D-05
X=0.27
          G 1= 0.2763D-01     GP 1=-0.1861D-01     GPP 1=-0.9799D-01     GPPP 1=-0.2867D+00
          G 3= 0.2270D-02     GP 3=-0.5066D-02     GPP 3= 0.2110D-02     GPPP 3= 0.4518D-01
          G 5=-0.2358D-06     GP 5=-0.1447D-05     GPP 5=-0.4084D-05     GPPP 5= 0.3328D-05
X=0.28
          G 1= 0.2744D-01     GP 1=-0.1961D-01     GPP 1=-0.1008D+00     GPPP 1=-0.2678D+00
          G 3= 0.2220D-02     GP 3=-0.5043D-02     GPP 3= 0.2533D-02     GPPP 3= 0.3958D-01
          G 5=-0.2504D-06     GP 5=-0.1488D-05     GPP 5=-0.4042D-05     GPPP 5= 0.5315D-05
X=0.29
          G 1= 0.2724D-01     GP 1=-0.2063D-01     GPP 1=-0.1033D+00     GPPP 1=-0.2470D+00
          G 3= 0.2170D-02     GP 3=-0.5016D-02     GPP 3= 0.2902D-02     GPPP 3= 0.3456D-01
          G 5=-0.2655D-06     GP 5=-0.1528D-05     GPP 5=-0.3978D-05     GPPP 5= 0.7482D-05
X=0.30
          G 1= 0.2702D-01     GP 1=-0.2167D-01     GPP 1=-0.1057D+00     GPPP 1=-0.2247D+00
          G 3= 0.2120D-02     GP 3=-0.4985D-02     GPP 3= 0.3224D-02     GPPP 3= 0.3006D-01
          G 5=-0.2810D-06     GP 5=-0.1567D-05     GPP 5=-0.3892D-05     GPPP 5= 0.9820D-05
X=0.31
          G 1= 0.2680D-01     GP 1=-0.2274D-01     GPP 1=-0.1078D+00     GPPP 1=-0.2009D+00
          G 3= 0.2070D-02     GP 3=-0.4951D-02     GPP 3= 0.3503D-02     GPPP 3= 0.2605D-01
          G 5=-0.2969D-06     GP 5=-0.1606D-05     GPP 5=-0.3782D-05     GPPP 5= 0.1232D-04
X=0.32
          G 1= 0.2657D-01     GP 1=-0.2383D-01     GPP 1=-0.1097D+00     GPPP 1=-0.1760D+00
          G 3= 0.2021D-02     GP 3=-0.4915D-02     GPP 3= 0.3745D-02     GPPP 3= 0.2248D-01
          G 5=-0.3131D-06     GP 5=-0.1643D-05     GPP 5=-0.3645D-05     GPPP 5= 0.1497D-04
X=0.33
          G 1= 0.2633D-01     GP 1=-0.2493D-01     GPP 1=-0.1114D+00     GPPP 1=-0.1501D+00
          G 3= 0.1972D-02     GP 3=-0.4877D-02     GPP 3= 0.3953D-02     GPPP 3= 0.1931D-01
          G 5=-0.3297D-06     GP 5=-0.1678D-05     GPP 5=-0.3482D-05     GPPP 5= 0.1777D-04
X=0.34
          G 1= 0.2607D-01     GP 1=-0.2605D-01     GPP 1=-0.1127D+00     GPPP 1=-0.1233D+00
          G 3= 0.1923D-02     GP 3=-0.4836D-02     GPP 3= 0.4131D-02     GPPP 3= 0.1652D-01
          G 5=-0.3467D-06     GP 5=-0.1712D-05     GPP 5=-0.3290D-05     GPPP 5= 0.2070D-04
X=0.35
          G 1= 0.2580D-01     GP 1=-0.2719D-01     GPP 1=-0.1138D+00     GPPP 1=-0.9575D-01
          G 3= 0.1875D-02     GP 3=-0.4794D-02     GPP 3= 0.4283D-02     GPPP 3= 0.1407D-01
          G 5=-0.3640D-06     GP 5=-0.1744D-05     GPP 5=-0.3068D-05     GPPP 5= 0.2375D-04
X=0.36
          G 1= 0.2553D-01     GP 1=-0.2833D-01     GPP 1=-0.1146D+00     GPPP 1=-0.6768D-01
          G 3= 0.1827D-02     GP 3=-0.4751D-02     GPP 3= 0.4413D-02     GPPP 3= 0.1193D-01
          G 5=-0.3816D-06     GP 5=-0.1773D-05     GPP 5=-0.2815D-05     GPPP 5= 0.2691D-04
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X=0.37
        G 1= 0.2524D-01    GP 1=-0.2948D-01    GPP 1=-0.1152D+00    GPPP 1=-0.3916D-01
        G 3= 0.1780D-02    GP 3=-0.4706D-02    GPP 3= 0.4522D-02    GPPP 3= 0.1007D-01
        G 5=-0.3994D-06    GP 5=-0.1800D-05    GPP 5=-0.2530D-05    GPPP 5= 0.3018D-04
X=0.38
        G 1= 0.2494D-01    GP 1=-0.3063D-01    GPP 1=-0.1154D+00    GPPP 1=-0.1030D-01
        G 3= 0.1733D-02    GP 3=-0.4660D-02    GPP 3= 0.4614D-02    GPPP 3= 0.8478D-02
        G 5=-0.4176D-06    GP 5=-0.1824D-05    GPP 5=-0.2212D-05    GPPP 5= 0.3353D-04
X=0.39
        G 1= 0.2463D-01    GP 1=-0.3178D-01    GPP 1=-0.1154D+00    GPPP 1= 0.1880D-01
        G 3= 0.1687D-02    GP 3=-0.4614D-02    GPP 3= 0.4691D-02    GPPP 3= 0.7115D-02
        G 5=-0.4359D-06    GP 5=-0.1844D-05    GPP 5=-0.1859D-05    GPPP 5= 0.3696D-04
X=0.40
        G 1= 0.2430D-01    GP 1=-0.3294D-01    GPP 1=-0.1150D+00    GPPP 1= 0.4804D-01
        G 3= 0.1641D-02    GP 3=-0.4567D-02    GPP 3= 0.4756D-02    GPPP 3= 0.5966D-02
        G 5=-0.4545D-06    GP 5=-0.1861D-05    GPP 5=-0.1472D-05    GPPP 5= 0.4045D-04
X=0.41
        G 1= 0.2397D-01    GP 1=-0.3408D-01    GPP 1=-0.1144D+00    GPPP 1= 0.7736D-01
        G 3= 0.1595D-02    GP 3=-0.4519D-02    GPP 3= 0.4811D-02    GPPP 3= 0.5010D-02
        G 5=-0.4731D-06    GP 5=-0.1874D-05    GPP 5=-0.1050D-05    GPPP 5= 0.4398D-04
X=0.42
        G 1= 0.2362D-01    GP 1=-0.3522D-01    GPP 1=-0.1135D+00    GPPP 1= 0.1067D+00
        G 3= 0.1550D-02    GP 3=-0.4471D-02    GPP 3= 0.4857D-02    GPPP 3= 0.4230D-02
        G 5=-0.4919D-06    GP 5=-0.1882D-05    GPP 5=-0.5928D-06    GPPP 5= 0.4754D-04
X=0.43
        G 1= 0.2326D-01    GP 1=-0.3635D-01    GPP 1=-0.1123D+00    GPPP 1= 0.1360D+00
        G 3= 0.1506D-02    GP 3=-0.4422D-02    GPP 3= 0.4895D-02    GPPP 3= 0.3609D-02
        G 5=-0.5108D-06    GP 5=-0.1885D-05    GPP 5=-0.9955D-07    GPPP 5= 0.5111D-04
X=0.44
        G 1= 0.2289D-01    GP 1=-0.3747D-01    GPP 1=-0.1108D+00    GPPP 1= 0.1651D+00
        G 3= 0.1462D-02    GP 3=-0.4373D-02    GPP 3= 0.4929D-02    GPPP 3= 0.3132D-02
        G 5=-0.5296D-06    GP 5=-0.1884D-05    GPP 5= 0.4293D-06    GPPP 5= 0.5466D-04
X=0.45
        G 1= 0.2251D-01    GP 1=-0.3857D-01    GPP 1=-0.1090D+00    GPPP 1= 0.1941D+00
        G 3= 0.1418D-02    GP 3=-0.4323D-02    GPP 3= 0.4958D-02    GPPP 3= 0.2785D-02
        G 5=-0.5484D-06    GP 5=-0.1876D-05    GPP 5= 0.9937D-06    GPPP 5= 0.5819D-04
X=0.46
        G 1= 0.2212D-01    GP 1=-0.3965D-01    GPP 1=-0.1069D+00    GPPP 1= 0.2230D+00
        G 3= 0.1376D-02    GP 3=-0.4273D-02    GPP 3= 0.4984D-02    GPPP 3= 0.2554D-02
        G 5=-0.5672D-06    GP 5=-0.1864D-05    GPP 5= 0.1593D-05    GPPP 5= 0.6167D-04
X=0.47
        G 1= 0.2172D-01    GP 1=-0.4070D-01    GPP 1=-0.1045D+00    GPPP 1= 0.2516D+00
        G 3= 0.1333D-02    GP 3=-0.4224D-02    GPP 3= 0.5009D-02    GPPP 3= 0.2430D-02
        G 5=-0.5857D-06    GP 5=-0.1844D-05    GPP 5= 0.2227D-05    GPPP 5= 0.6507D-04
X=0.48
        G 1= 0.2131D-01    GP 1=-0.4174D-01    GPP 1=-0.1019D+00    GPPP 1= 0.2799D+00
        G 3= 0.1291D-02    GP 3=-0.4173D-02    GPP 3= 0.5033D-02    GPPP 3= 0.2401D-02
        G 5=-0.6040D-06    GP 5=-0.1819D-05    GPP 5= 0.2895D-05    GPPP 5= 0.6838D-04
X=0.49
        G 1= 0.2089D-01    GP 1=-0.4274D-01    GPP 1=-0.9892D-01    GPPP 1= 0.3080D+00
        G 3= 0.1250D-02    GP 3=-0.4123D-02    GPP 3= 0.5057D-02    GPPP 3= 0.2458D-02
        G 5=-0.6221D-06    GP 5=-0.1786D-05    GPP 5= 0.3595D-05    GPPP 5= 0.7157D-04
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X=0.50
          G 1= 0.2045D-01     GP 1=-0.4371D-01     GPP 1=-0.9571D-01     GPPP 1= 0.3357D+00
          G 3= 0.1209D-02     GP 3=-0.4072D-02     GPP 3= 0.5082D-02     GPPP 3= 0.2593D-02
          G 5=-0.6398D-06     GP 5=-0.1747D-05     GPP 5= 0.4326D-05     GPPP 5= 0.7460D-04
X=0.51
          G 1= 0.2001D-01     GP 1=-0.4465D-01     GPP 1=-0.9221D-01     GPPP 1= 0.3632D+00
          G 3= 0.1168D-02     GP 3=-0.4021D-02     GPP 3= 0.5109D-02     GPPP 3= 0.2797D-02
          G 5=-0.6570D-06     GP 5=-0.1700D-05     GPP 5= 0.5087D-05     GPPP 5= 0.7746D-04
X=0.52
          G 1= 0.1956D-01     GP 1=-0.4556D-01     GPP 1=-0.8844D-01     GPPP 1= 0.3904D+00
          G 3= 0.1128D-02     GP 3=-0.3970D-02     GPP 3= 0.5138D-02     GPPP 3= 0.3063D-02
          G 5=-0.6738D-06     GP 5=-0.1645D-05     GPP 5= 0.5875D-05     GPPP 5= 0.8010D-04
X=0.53
          G 1= 0.1910D-01     GP 1=-0.4642D-01     GPP 1=-0.8440D-01     GPPP 1= 0.4172D+00
          G 3= 0.1089D-02     GP 3=-0.3918D-02     GPP 3= 0.5170D-02     GPPP 3= 0.3386D-02
          G 5=-0.6899D-06     GP 5=-0.1582D-05     GPP 5= 0.6689D-05     GPPP 5= 0.8251D-04
X=0.54
          G 1= 0.1863D-01     GP 1=-0.4724D-01     GPP 1=-0.8010D-01     GPPP 1= 0.4437D+00
          G 3= 0.1050D-02     GP 3=-0.3867D-02     GPP 3= 0.5206D-02     GPPP 3= 0.3760D-02
          G 5=-0.7054D-06     GP 5=-0.1511D-05     GPP 5= 0.7525D-05     GPPP 5= 0.8465D-04
X=0.55
          G 1= 0.1815D-01     GP 1=-0.4802D-01     GPP 1=-0.7553D-01     GPPP 1= 0.4698D+00
          G 3= 0.1011D-02     GP 3=-0.3814D-02     GPP 3= 0.5245D-02     GPPP 3= 0.4180D-02
          G 5=-0.7201D-06     GP 5=-0.1431D-05     GPP 5= 0.8382D-05     GPPP 5= 0.8649D-04
X=0.56
          G 1= 0.1767D-01     GP 1=-0.4875D-01     GPP 1=-0.7070D-01     GPPP 1= 0.4957D+00
          G 3= 0.9735D-03     GP 3=-0.3762D-02     GPP 3= 0.5289D-02     GPPP 3= 0.4640D-02
          G 5=-0.7340D-06     GP 5=-0.1343D-05     GPP 5= 0.9255D-05     GPPP 5= 0.8798D-04
X=0.57
          G 1= 0.1718D-01     GP 1=-0.4943D-01     GPP 1=-0.6562D-01     GPPP 1= 0.5211D+00
          G 3= 0.9361D-03     GP 3=-0.3708D-02     GPP 3= 0.5338D-02     GPPP 3= 0.5138D-02
          G 5=-0.7470D-06     GP 5=-0.1246D-05     GPP 5= 0.1014D-04     GPPP 5= 0.8909D-04
X=0.58
          G 1= 0.1668D-01     GP 1=-0.5006D-01     GPP 1=-0.6028D-01     GPPP 1= 0.5463D+00
          G 3= 0.8993D-03     GP 3=-0.3655D-02     GPP 3= 0.5392D-02     GPPP 3= 0.5668D-02
          G 5=-0.7590D-06     GP 5=-0.1140D-05     GPP 5= 0.1104D-04     GPPP 5= 0.8978D-04
X=0.59
          G 1= 0.1618D-01     GP 1=-0.5064D-01     GPP 1=-0.5469D-01     GPPP 1= 0.5711D+00
          G 3= 0.8630D-03     GP 3=-0.3601D-02     GPP 3= 0.5452D-02     GPPP 3= 0.6229D-02
          G 5=-0.7698D-06     GP 5=-0.1026D-05     GPP 5= 0.1194D-04     GPPP 5= 0.9002D-04
X=0.60
          G 1= 0.1567D-01     GP 1=-0.5115D-01     GPP 1=-0.4886D-01     GPPP 1= 0.5955D+00
          G 3= 0.8273D-03     GP 3=-0.3546D-02     GPP 3= 0.5517D-02     GPPP 3= 0.6816D-02
          G 5=-0.7795D-06     GP 5=-0.9017D-06     GPP 5= 0.1284D-04     GPPP 5= 0.8975D-04
X=0.61
          G 1= 0.1516D-01     GP 1=-0.5161D-01     GPP 1=-0.4278D-01     GPPP 1= 0.6197D+00
          G 3= 0.7921D-03     GP 3=-0.3490D-02     GPP 3= 0.5588D-02     GPPP 3= 0.7428D-02
          G 5=-0.7879D-06     GP 5=-0.7688D-06     GPP 5= 0.1373D-04     GPPP 5= 0.8893D-04
X=0.62
          G 1= 0.1464D-01     GP 1=-0.5201D-01     GPP 1=-0.3646D-01     GPPP 1= 0.6435D+00
          G 3= 0.7575D-03     GP 3=-0.3434D-02     GPP 3= 0.5665D-02     GPPP 3= 0.8062D-02
          G 5=-0.7949D-06     GP 5=-0.6271D-06     GPP 5= 0.1462D-04     GPPP 5= 0.8752D-04
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X=0.63
          G 1= 0.1412D-01     GP 1=-0.5234D-01     GPP 1=-0.2991D-01     GPPP 1= 0.6670D+00
          G 3= 0.7235D-03     GP 3=-0.3377D-02     GPP 3= 0.5749D-02     GPPP 3= 0.8716D-02
          G 5=-0.8004D-06     GP 5=-0.4766D-06     GPP 5= 0.1548D-04     GPPP 5= 0.8547D-04
X=0.64
          G 1= 0.1359D-01     GP 1=-0.5261D-01     GPP 1=-0.2312D-01     GPPP 1= 0.6902D+00
          G 3= 0.6900D-03     GP 3=-0.3319D-02     GPP 3= 0.5840D-02     GPPP 3= 0.9389D-02
          G 5=-0.8044D-06     GP 5=-0.3175D-06     GPP 5= 0.1633D-04     GPPP 5= 0.8273D-04
X=0.65
          G 1= 0.1306D-01     GP 1=-0.5280D-01     GPP 1=-0.1611D-01     GPPP 1= 0.7131D+00
          G 3= 0.6571D-03     GP 3=-0.3260D-02     GPP 3= 0.5937D-02     GPPP 3= 0.1008D-01
          G 5=-0.8067D-06     GP 5=-0.1502D-06     GPP 5= 0.1714D-04     GPPP 5= 0.7925D-04
X=0.66
          G 1= 0.1253D-01     GP 1=-0.5293D-01     GPP 1=-0.8863D-02     GPPP 1= 0.7357D+00
          G 3= 0.6248D-03     GP 3=-0.3200D-02     GPP 3= 0.6041D-02     GPPP 3= 0.1078D-01
          G 5=-0.8074D-06     GP 5= 0.2501D-07     GPP 5= 0.1791D-04     GPPP 5= 0.7496D-04
X=0.67
          G 1= 0.1200D-01     GP 1=-0.5298D-01     GPP 1=-0.1394D-02     GPPP 1= 0.7580D+00
          G 3= 0.5931D-03     GP 3=-0.3139D-02     GPP 3= 0.6153D-02     GPPP 3= 0.1150D-01
          G 5=-0.8062D-06     GP 5= 0.2077D-06     GPP 5= 0.1864D-04     GPPP 5= 0.6982D-04
X=0.68
          G 1= 0.1147D-01     GP 1=-0.5295D-01     GPP 1= 0.6297D-02     GPPP 1= 0.7800D+00
          G 3= 0.5620D-03     GP 3=-0.3077D-02     GPP 3= 0.6271D-02     GPPP 3= 0.1223D-01
          G 5=-0.8032D-06     GP 5= 0.3975D-06     GPP 5= 0.1931D-04     GPPP 5= 0.6378D-04
X=0.69
          G 1= 0.1095D-01     GP 1=-0.5285D-01     GPP 1= 0.1421D-01     GPPP 1= 0.8018D+00
          G 3= 0.5315D-03     GP 3=-0.3014D-02     GPP 3= 0.6397D-02     GPPP 3= 0.1297D-01
          G 5=-0.7983D-06     GP 5= 0.5935D-06     GPP 5= 0.1991D-04     GPPP 5= 0.5675D-04
X=0.70
          G 1= 0.1042D-01     GP 1=-0.5267D-01     GPP 1= 0.2233D-01     GPPP 1= 0.8233D+00
          G 3= 0.5017D-03     GP 3=-0.2949D-02     GPP 3= 0.6531D-02     GPPP 3= 0.1373D-01
          G 5=-0.7914D-06     GP 5= 0.7953D-06     GPP 5= 0.2044D-04     GPPP 5= 0.4870D-04
X=0.71
          G 1= 0.9892D-02     GP 1=-0.5240D-01     GPP 1= 0.3067D-01     GPPP 1= 0.8445D+00
          G 3= 0.4725D-03     GP 3=-0.2883D-02     GPP 3= 0.6672D-02     GPPP 3= 0.1449D-01
          G 5=-0.7824D-06     GP 5= 0.1002D-05     GPP 5= 0.2089D-04     GPPP 5= 0.3955D-04
X=0.72
          G 1= 0.9370D-02     GP 1=-0.5205D-01     GPP 1= 0.3922D-01     GPPP 1= 0.8655D+00
          G 3= 0.4440D-03     GP 3=-0.2816D-02     GPP 3= 0.6821D-02     GPPP 3= 0.1527D-01
          G 5=-0.7713D-06     GP 5= 0.1213D-05     GPP 5= 0.2123D-04     GPPP 5= 0.2923D-04
X=0.73
          G 1= 0.8851D-02     GP 1=-0.5162D-01     GPP 1= 0.4798D-01     GPPP 1= 0.8863D+00
          G 3= 0.4162D-03     GP 3=-0.2747D-02     GPP 3= 0.6977D-02     GPPP 3= 0.1605D-01
          G 5=-0.7581D-06     GP 5= 0.1426D-05     GPP 5= 0.2147D-04     GPPP 5= 0.1768D-04
X=0.74
          G 1= 0.8337D-02     GP 1=-0.5109D-01     GPP 1= 0.5695D-01     GPPP 1= 0.9068D+00
          G 3= 0.3891D-03     GP 3=-0.2676D-02     GPP 3= 0.7142D-02     GPPP 3= 0.1685D-01
          G 5=-0.7428D-06     GP 5= 0.1641D-05     GPP 5= 0.2159D-04     GPPP 5= 0.4824D-05
X=0.75
          G 1= 0.7829D-02     GP 1=-0.5048D-01     GPP 1= 0.6612D-01     GPPP 1= 0.9272D+00
          G 3= 0.3627D-03     GP 3=-0.2604D-02     GPP 3= 0.7314D-02     GPPP 3= 0.1765D-01
          G 5=-0.7253D-06     GP 5= 0.1857D-05     GPP 5= 0.2157D-04     GPPP 5=-0.9407D-05
```

```
X=0.76
          G 1= 0.7328D-02     GP 1=-0.4977D-01     GPP 1= 0.7549D-01     GPPP 1= 0.9473D+00
          G 3= 0.3370D-03     GP 3=-0.2530D-02     GPP 3= 0.7495D-02     GPPP 3= 0.1846D-01
          G 5=-0.7057D-06     GP 5= 0.2072D-05     GPP 5= 0.2140D-04     GPPP 5=-0.2509D-04
X=0.77
          G 1= 0.6834D-02     GP 1=-0.4897D-01     GPP 1= 0.8506D-01     GPPP 1= 0.9672D+00
          G 3= 0.3121D-03     GP 3=-0.2454D-02     GPP 3= 0.7683D-02     GPPP 3= 0.1928D-01
          G 5=-0.6839D-06     GP 5= 0.2284D-05     GPP 5= 0.2107D-04     GPPP 5=-0.4230D-04
X=0.78
          G 1= 0.6348D-02     GP 1=-0.4807D-01     GPP 1= 0.9484D-01     GPPP 1= 0.9868D+00
          G 3= 0.2880D-03     GP 3=-0.2376D-02     GPP 3= 0.7880D-02     GPPP 3= 0.2011D-01
          G 5=-0.6600D-06     GP 5= 0.2492D-05     GPP 5= 0.2055D-04     GPPP 5=-0.6113D-04
X=0.79
          G 1= 0.5873D-02     GP 1=-0.4707D-01     GPP 1= 0.1048D+00     GPPP 1= 0.1006D+01
          G 3= 0.2646D-03     GP 3=-0.2296D-02     GPP 3= 0.8086D-02     GPPP 3= 0.2094D-01
          G 5=-0.6340D-06     GP 5= 0.2694D-05     GPP 5= 0.1984D-04     GPPP 5=-0.8164D-04
X=0.80
          G 1= 0.5407D-02     GP 1=-0.4597D-01     GPP 1= 0.1150D+00     GPPP 1= 0.1026D+01
          G 3= 0.2420D-03     GP 3=-0.2214D-02     GPP 3= 0.8299D-02     GPPP 3= 0.2179D-01
          G 5=-0.6061D-06     GP 5= 0.2888D-05     GPP 5= 0.1892D-04     GPPP 5=-0.1039D-03
X=0.81
          G 1= 0.4953D-02     GP 1=-0.4477D-01     GPP 1= 0.1253D+00     GPPP 1= 0.1045D+01
          G 3= 0.2203D-03     GP 3=-0.2130D-02     GPP 3= 0.8521D-02     GPPP 3= 0.2264D-01
          G 5=-0.5763D-06     GP 5= 0.3072D-05     GPP 5= 0.1776D-04     GPPP 5=-0.1281D-03
X=0.82
          G 1= 0.4512D-02     GP 1=-0.4346D-01     GPP 1= 0.1359D+00     GPPP 1= 0.1064D+01
          G 3= 0.1994D-03     GP 3=-0.2044D-02     GPP 3= 0.8752D-02     GPPP 3= 0.2350D-01
          G 5=-0.5447D-06     GP 5= 0.3242D-05     GPP 5= 0.1636D-04     GPPP 5=-0.1542D-03
X=0.83
          G 1= 0.4084D-02     GP 1=-0.4205D-01     GPP 1= 0.1466D+00     GPPP 1= 0.1083D+01
          G 3= 0.1794D-03     GP 3=-0.1955D-02     GPP 3= 0.8991D-02     GPPP 3= 0.2437D-01
          G 5=-0.5114D-06     GP 5= 0.3397D-05     GPP 5= 0.1468D-04     GPPP 5=-0.1823D-03
X=0.84
          G 1= 0.3671D-02     GP 1=-0.4053D-01     GPP 1= 0.1575D+00     GPPP 1= 0.1101D+01
          G 3= 0.1603D-03     GP 3=-0.1864D-02     GPP 3= 0.9239D-02     GPPP 3= 0.2524D-01
          G 5=-0.4767D-06     GP 5= 0.3534D-05     GPP 5= 0.1271D-04     GPPP 5=-0.2126D-03
X=0.85
          G 1= 0.3273D-02     GP 1=-0.3890D-01     GPP 1= 0.1686D+00     GPPP 1= 0.1120D+01
          G 3= 0.1422D-03     GP 3=-0.1770D-02     GPP 3= 0.9496D-02     GPPP 3= 0.2613D-01
          G 5=-0.4407D-06     GP 5= 0.3650D-05     GPP 5= 0.1043D-04     GPPP 5=-0.2451D-03
X=0.86
          G 1= 0.2893D-02     GP 1=-0.3716D-01     GPP 1= 0.1799D+00     GPPP 1= 0.1138D+01
          G 3= 0.1249D-03     GP 3=-0.1674D-02     GPP 3= 0.9762D-02     GPPP 3= 0.2702D-01
          G 5=-0.4037D-06     GP 5= 0.3741D-05     GPP 5= 0.7808D-05     GPPP 5=-0.2800D-03
X=0.87
          G 1= 0.2530D-02     GP 1=-0.3530D-01     GPP 1= 0.1914D+00     GPPP 1= 0.1156D+01
          G 3= 0.1087D-03     GP 3=-0.1575D-02     GPP 3= 0.1004D-01     GPPP 3= 0.2792D-01
          G 5=-0.3659D-06     GP 5= 0.3804D-05     GPP 5= 0.4828D-05     GPPP 5=-0.3173D-03
X=0.88
          G 1= 0.2187D-02     GP 1=-0.3333D-01     GPP 1= 0.2030D+00     GPPP 1= 0.1174D+01
          G 3= 0.9343D-04     GP 3=-0.1473D-02     GPP 3= 0.1032D-01     GPPP 3= 0.2883D-01
          G 5=-0.3276D-06     GP 5= 0.3836D-05     GPP 5= 0.1462D-05     GPPP 5=-0.3571D-03
```

```
X=0.89
        G 1= 0.1863D-02     GP 1=-0.3124D-01     GPP 1= 0.2149D+00     GPPP 1= 0.1192D+01
        G 3= 0.7921D-04     GP 3=-0.1369D-02     GPP 3= 0.1061D-01     GPPP 3= 0.2975D-01
        G 5=-0.2892D-06     GP 5= 0.3832D-05     GPP 5=-0.2315D-05     GPPP 5=-0.3996D-03
X=0.90
        G 1= 0.1562D-02     GP 1=-0.2903D-01     GPP 1= 0.2269D+00     GPPP 1= 0.1210D+01
        G 3= 0.6606D-04     GP 3=-0.1261D-02     GPP 3= 0.1092D-01     GPPP 3= 0.3067D-01
        G 5=-0.2510D-06     GP 5= 0.3787D-05     GPP 5=-0.6531D-05     GPPP 5=-0.4449D-03
X=0.91
        G 1= 0.1283D-02     GP 1=-0.2670D-01     GPP 1= 0.2391D+00     GPPP 1= 0.1228D+01
        G 3= 0.5400D-04     GP 3=-0.1150D-02     GPP 3= 0.1123D-01     GPPP 3= 0.3161D-01
        G 5=-0.2134D-06     GP 5= 0.3699D-05     GPP 5=-0.1121D-04     GPPP 5=-0.4930D-03
X=0.92
        G 1= 0.1028D-02     GP 1=-0.2425D-01     GPP 1= 0.2514D+00     GPPP 1= 0.1245D+01
        G 3= 0.4306D-04     GP 3=-0.1036D-02     GPP 3= 0.1155D-01     GPPP 3= 0.3256D-01
        G 5=-0.1770D-06     GP 5= 0.3561D-05     GPP 5=-0.1639D-04     GPPP 5=-0.5441D-03
X=0.93
        G 1= 0.7979D-03     GP 1=-0.2167D-01     GPP 1= 0.2640D+00     GPPP 1= 0.1263D+01
        G 3= 0.3327D-04     GP 3=-0.9192D-03     GPP 3= 0.1188D-01     GPPP 3= 0.3351D-01
        G 5=-0.1422D-06     GP 5= 0.3368D-05     GPP 5=-0.2210D-04     GPPP 5=-0.5983D-03
X=0.94
        G 1= 0.5944D-03     GP 1=-0.1897D-01     GPP 1= 0.2767D+00     GPPP 1= 0.1280D+01
        G 3= 0.2467D-04     GP 3=-0.7987D-03     GPP 3= 0.1222D-01     GPPP 3= 0.3447D-01
        G 5=-0.1097D-06     GP 5= 0.3116D-05     GPP 5=-0.2836D-04     GPPP 5=-0.6558D-03
X=0.95
        G 1= 0.4185D-03     GP 1=-0.1614D-01     GPP 1= 0.2896D+00     GPPP 1= 0.1297D+01
        G 3= 0.1730D-04     GP 3=-0.6748D-03     GPP 3= 0.1257D-01     GPPP 3= 0.3545D-01
        G 5=-0.7992D-07     GP 5= 0.2798D-05     GPP 5=-0.3521D-04     GPPP 5=-0.7165D-03
X=0.96
        G 1= 0.2716D-03     GP 1=-0.1318D-01     GPP 1= 0.3026D+00     GPPP 1= 0.1315D+01
        G 3= 0.1118D-04     GP 3=-0.5473D-03     GPP 3= 0.1293D-01     GPPP 3= 0.3643D-01
        G 5=-0.5370D-07     GP 5= 0.2409D-05     GPP 5=-0.4269D-04     GPPP 5=-0.7808D-03
X=0.97
        G 1= 0.1550D-03     GP 1=-0.1008D-01     GPP 1= 0.3159D+00     GPPP 1= 0.1332D+01
        G 3= 0.6350D-05     GP 3=-0.4162D-03     GPP 3= 0.1330D-01     GPPP 3= 0.3743D-01
        G 5=-0.3175D-07     GP 5= 0.1941D-05     GPP 5=-0.5083D-04     GPPP 5=-0.8487D-03
X=0.98
        G 1= 0.6994D-04     GP 1=-0.6858D-02     GPP 1= 0.3293D+00     GPPP 1= 0.1349D+01
        G 3= 0.2853D-05     GP 3=-0.2813D-03     GPP 3= 0.1368D-01     GPPP 3= 0.3843D-01
        G 5=-0.1488D-07     GP 5= 0.1389D-05     GPP 5=-0.5966D-04     GPPP 5=-0.9203D-03
X=0.99
        G 1= 0.1783D-04     GP 1=-0.3497D-02     GPP 1= 0.3428D+00     GPPP 1= 0.1366D+01
        G 3= 0.7232D-06     GP 3=-0.1426D-03     GPP 3= 0.1406D-01     GPPP 3= 0.3945D-01
        G 5=-0.3979D-08     GP 5= 0.7441D-06     GPP 5=-0.6923D-04     GPPP 5=-0.9958D-03
X=1.00
        G 1= 0.0             GP 1= 0.0            GPP 1= 0.3566D+00     GPPP 1= 0.1383D+01
        G 3= 0.0             GP 3= 0.0            GPP 3= 0.1446D-01     GPPP 3= 0.4045D-01
        G 5= 0.0             GP 5= 0.0            GPP 5=-0.7958D-04     GPPP 5=-0.1067D-02
  IER=  0
```

```
                    DIMENSIONLESS  STREAM FUNCTION  PSI(X,MU)
X=0.0
          MU=0.0      PSI=    0.25000D-01
          MU=0.10     PSI=    0.24997D-01
          MU=0.20     PSI=    0.24960D-01
          MU=0.30     PSI=    0.24797D-01
          MU=0.40     PSI=    0.24360D-01
          MU=0.50     PSI=    0.23437D-01
          MU=0.60     PSI=    0.21760D-01
          MU=0.70     PSI=    0.18997D-01
          MU=0.80     PSI=    0.14760D-01
          MU=0.90     PSI=    0.85975D-02
          MU=1.00     PSI=    0.0
X=0.10
          MU=0.0      PSI=    0.24994D-01
          MU=0.10     PSI=    0.24974D-01
          MU=0.20     PSI=    0.24886D-01
          MU=0.30     PSI=    0.24646D-01
          MU=0.40     PSI=    0.24116D-01
          MU=0.50     PSI=    0.23099D-01
          MU=0.60     PSI=    0.21346D-01
          MU=0.70     PSI=    0.18550D-01
          MU=0.80     PSI=    0.14348D-01
          MU=0.90     PSI=    0.83224D-02
          MU=1.00     PSI=    0.0
X=0.20
          MU=0.0      PSI=    0.24767D-01
          MU=0.10     PSI=    0.24715D-01
          MU=0.20     PSI=    0.24533D-01
          MU=0.30     PSI=    0.24152D-01
          MU=0.40     PSI=    0.23453D-01
          MU=0.50     PSI=    0.22270D-01
          MU=0.60     PSI=    0.20391D-01
          MU=0.70     PSI=    0.17555D-01
          MU=0.80     PSI=    0.13456D-01
          MU=0.90     PSI=    0.77380D-02
          MU=1.00     PSI=    0.0
X=0.30
          MU=0.0      PSI=    0.23844D-01
          MU=0.10     PSI=    0.23763D-01
          MU=0.20     PSI=    0.23501D-01
          MU=0.30     PSI=    0.23001D-01
          MU=0.40     PSI=    0.22167D-01
          MU=0.50     PSI=    0.20865D-01
          MU=0.60     PSI=    0.18924D-01
          MU=0.70     PSI=    0.16134D-01
          MU=0.80     PSI=    0.12247D-01
          MU=0.90     PSI=    0.69767D-02
          MU=1.00     PSI=    0.0
X=0.40
          MU=0.0      PSI=    0.21840D-01
          MU=0.10     PSI=    0.21743D-01
          MU=0.20     PSI=    0.21439D-01
          MU=0.30     PSI=    0.20883D-01
          MU=0.40     PSI=    0.20000D-01
          MU=0.50     PSI=    0.18689D-01
          MU=0.60     PSI=    0.16814D-01
          MU=0.70     PSI=    0.14214D-01
          MU=0.80     PSI=    0.10698D-01
          MU=0.90     PSI=    0.60430D-02
          MU=1.00     PSI=    0.0
X=0.50
          MU=0.0      PSI=    0.18639D-01
          MU=0.10     PSI=    0.18542D-01
          MU=0.20     PSI=    0.18242D-01
          MU=0.30     PSI=    0.17705D-01
          MU=0.40     PSI=    0.16877D-01
          MU=0.50     PSI=    0.15681D-01
          MU=0.60     PSI=    0.14019D-01
          MU=0.70     PSI=    0.11772D-01
          MU=0.80     PSI=    0.87985D-02
          MU=0.90     PSI=    0.49358D-02
          MU=1.00     PSI=    0.0
X=0.60
          MU=0.0      PSI=    0.14427D-01
          MU=0.10     PSI=    0.14345D-01
          MU=0.20     PSI=    0.14089D-01
          MU=0.30     PSI=    0.13638D-01
          MU=0.40     PSI=    0.12955D-01
          MU=0.50     PSI=    0.11986D-01
          MU=0.60     PSI=    0.10665D-01
          MU=0.70     PSI=    0.89098D-02
          MU=0.80     PSI=    0.66236D-02
          MU=0.90     PSI=    0.36954D-02
          MU=1.00     PSI=    0.0
X=0.70
          MU=0.0      PSI=    0.96638D-02
          MU=0.10     PSI=    0.96046D-02
          MU=0.20     PSI=    0.94225D-02
          MU=0.30     PSI=    0.91037D-02
          MU=0.40     PSI=    0.86254D-02
          MU=0.50     PSI=    0.79558D-02
          MU=0.60     PSI=    0.70540D-02
          MU=0.70     PSI=    0.58702D-02
          MU=0.80     PSI=    0.43461D-02
          MU=0.90     PSI=    0.24145D-02
          MU=1.00     PSI=    0.0
```

```
X=0.80                                                  X=1.00
                MU=0.0      PSI=   0.50429D-02                          MU=0.0      PSI=   0.0
                MU=0.10     PSI=   0.50106D-02                          MU=0.10     PSI=   0.0
                MU=0.20     PSI=   0.49114D-02                          MU=0.20     PSI=   0.0
                MU=0.30     PSI=   0.47388D-02                          MU=0.30     PSI=   0.0
                MU=0.40     PSI=   0.44816D-02                          MU=0.40     PSI=   0.0
                MU=0.50     PSI=   0.41244D-02                          MU=0.50     PSI=   0.0
                MU=0.60     PSI=   0.36474D-02                          MU=0.60     PSI=   0.0
                MU=0.70     PSI=   0.30266D-02                          MU=0.70     PSI=   0.0
                MU=0.80     PSI=   0.22338D-02                          MU=0.80     PSI=   0.0
                MU=0.90     PSI=   0.12370D-02                          MU=0.90     PSI=   0.0
                MU=1.00     PSI=   0.0                                  MU=1.00     PSI=   0.0
X=0.90
                MU=0.0      PSI=   0.14623D-02
                MU=0.10     PSI=   0.14526D-02
                MU=0.20     PSI=   0.14230D-02
                MU=0.30     PSI=   0.13717D-02
                MU=0.40     PSI=   0.12956D-02
                MU=0.50     PSI=   0.11904D-02
                MU=0.60     PSI=   0.10507D-02
                MU=0.70     PSI=   0.87000D-03
                MU=0.80     PSI=   0.64063D-03
                MU=0.90     PSI=   0.35386D-03
                MU=1.00     PSI=   0.0
```

```
                    SPEED DISTRIBUTION V*C**2/(NU*D)
X=0.0
          MU=0.0      W=    0.0
          MU=0.10     W=    0.40000D-02
          MU=0.20     W=    0.16000D-01
          MU=0.30     W=    0.36000D-01
          MU=0.40     W=    0.64000D-01
          MU=0.50     W=    0.10000D+00
          MU=0.60     W=    0.14400D+00
          MU=0.70     W=    0.19600D+00
          MU=0.80     W=    0.25600D+00
          MU=0.90     W=    0.32400D+00
          MU=1.00     W=    0.40000D+00
X=0.10
          MU=0.0      W=    0.47140D-02
          MU=0.10     W=    0.10431D-01
          MU=0.20     W=    0.23239D-01
          MU=0.30     W=    0.42336D-01
          MU=0.40     W=    0.68111D-01
          MU=0.50     W=    0.10081D+00
          MU=0.60     W=    0.14056D+00
          MU=0.70     W=    0.18742D+00
          MU=0.80     W=    0.24141D+00
          MU=0.90     W=    0.30256D+00
          MU=1.00     W=    0.37087D+00
X=0.20
          MU=0.0      W=    0.24721D-01
          MU=0.10     W=    0.27509D-01
          MU=0.20     W=    0.35886D-01
          MU=0.30     W=    0.49892D-01
          MU=0.40     W=    0.69563D-01
          MU=0.50     W=    0.94912D-01
          MU=0.60     W=    0.12594D+00
          MU=0.70     W=    0.16263D+00
          MU=0.80     W=    0.20500D+00
          MU=0.90     W=    0.25301D+00
          MU=1.00     W=    0.30668D+00
X=0.30
          MU=0.0      W=    0.47326D-01
          MU=0.10     W=    0.48571D-01
          MU=0.20     W=    0.52567D-01
          MU=0.30     W=    0.59940D-01
          MU=0.40     W=    0.71338D-01
          MU=0.50     W=    0.87166D-01
          MU=0.60     W=    0.10756D+00
          MU=0.70     W=    0.13250D+00
          MU=0.80     W=    0.16190D+00
          MU=0.90     W=    0.19567D+00
          MU=1.00     W=    0.23375D+00
X=0.40
          MU=0.0      W=    0.65226D-01
          MU=0.10     W=    0.65680D-01
          MU=0.20     W=    0.67166D-01
          MU=0.30     W=    0.70042D-01
          MU=0.40     W=    0.74835D-01
          MU=0.50     W=    0.82118D-01
          MU=0.60     W=    0.92369D-01
          MU=0.70     W=    0.10588D+00
          MU=0.80     W=    0.12277D+00
          MU=0.90     W=    0.14301D+00
          MU=1.00     W=    0.16653D+00
X=0.50
          MU=0.0      W=    0.75215D-01
          MU=0.10     W=    0.75270D-01
          MU=0.20     W=    0.75476D-01
          MU=0.30     W=    0.75954D-01
          MU=0.40     W=    0.76912D-01
          MU=0.50     W=    0.78642D-01
          MU=0.60     W=    0.81498D-01
          MU=0.70     W=    0.85853D-01
          MU=0.80     W=    0.92041D-01
          MU=0.90     W=    0.10030D+00
          MU=1.00     W=    0.11078D+00
X=0.60
          MU=0.0      W=    0.76397D-01
          MU=0.10     W=    0.76272D-01
          MU=0.20     W=    0.75901D-01
          MU=0.30     W=    0.75287D-01
          MU=0.40     W=    0.74444D-01
          MU=0.50     W=    0.73397D-01
          MU=0.60     W=    0.72191D-01
          MU=0.70     W=    0.70897D-01
          MU=0.80     W=    0.69613D-01
          MU=0.90     W=    0.68466D-01
          MU=1.00     W=    0.67612D-01
X=0.70
          MU=0.0      W=    0.68920D-01
          MU=0.10     W=    0.68742D-01
          MU=0.20     W=    0.68197D-01
          MU=0.30     W=    0.67246D-01
          MU=0.40     W=    0.65822D-01
          MU=0.50     W=    0.63826D-01
          MU=0.60     W=    0.61118D-01
          MU=0.70     W=    0.57489D-01
          MU=0.80     W=    0.52619D-01
          MU=0.90     W=    0.45946D-01
          MU=1.00     W=    0.36260D-01
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X=0.80                                                   X=1.00
                 MU=0.0      W=    0.53305D-01                           MU=0.0      W=    0.0
                 MU=0.10     W=    0.53149D-01                           MU=0.10     W=    0.0
                 MU=0.20     W=    0.52665D-01                           MU=0.20     W=    0.0
                 MU=0.30     W=    0.51812D-01                           MU=0.30     W=    0.0
                 MU=0.40     W=    0.50512D-01                           MU=0.40     W=    0.0
                 MU=0.50     W=    0.48646D-01                           MU=0.50     W=    0.0
                 MU=0.60     W=    0.46037D-01                           MU=0.60     W=    0.0
                 MU=0.70     W=    0.42403D-01                           MU=0.70     W=    0.0
                 MU=0.80     W=    0.37259D-01                           MU=0.80     W=    0.0
                 MU=0.90     W=    0.29560D-01                           MU=0.90     W=    0.0
                 MU=1.00     W=    0.15394D-01                           MU=1.00     W=    0.0
X=0.90
                 MU=0.0      W=    0.30147D-01
                 MU=0.10     W=    0.30056D-01
                 MU=0.20     W=    0.29774D-01
                 MU=0.30     W=    0.29273D-01
                 MU=0.40     W=    0.28505D-01
                 MU=0.50     W=    0.27391D-01
                 MU=0.60     W=    0.25816D-01
                 MU=0.70     W=    0.23591D-01
                 MU=0.80     W=    0.20376D-01
                 MU=0.90     W=    0.15375D-01
                 MU=1.00     W=    0.36876D-02
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                                      PRESSURE DISTRIBUTION
X=0.10                                    X=0.50
            MU=0.0    P=   0.0                        MU=0.0    P=   0.0
            MU=0.10   P=   0.19322D-01                MU=0.10   P=   0.14328D+00
            MU=0.20   P=   0.28739D+00                MU=0.20   P=   0.28757D+00
            MU=0.30   P=   0.67035D+00                MU=0.30   P=   0.43370D+00
            MU=0.40   P=   0.10726D+01                MU=0.40   P=   0.58221D+00
            MU=0.50   P=   0.14696D+01                MU=0.50   P=   0.73334D+00
            MU=0.60   P=   0.18581D+01                MU=0.60   P=   0.88706D+00
            MU=0.70   P=   0.22393D+01                MU=0.70   P=   0.10432D+01
            MU=0.80   P=   0.26147D+01                MU=0.80   P=   0.12014D+01
            MU=0.90   P=   0.29858D+01                MU=0.90   P=   0.13613D+01
            MU=1.00   P=   0.33537D+01                MU=1.00   P=   0.15228D+01
X=0.20                                    X=0.60
            MU=0.0    P=   0.0                        MU=0.0    P=   0.0
            MU=0.10   P=   0.17187D+00                MU=0.10   P=   0.13732D+00
            MU=0.20   P=   0.37290D+00                MU=0.20   P=   0.27519D+00
            MU=0.30   P=   0.60893D+00                MU=0.30   P=   0.41410D+00
            MU=0.40   P=   0.86979D+00                MU=0.40   P=   0.55443D+00
            MU=0.50   P=   0.11440D+01                MU=0.50   P=   0.69643D+00
            MU=0.60   P=   0.14241D+01                MU=0.60   P=   0.84020D+00
            MU=0.70   P=   0.17062D+01                MU=0.70   P=   0.98576D+00
            MU=0.80   P=   0.19883D+01                MU=0.80   P=   0.11330D+01
            MU=0.90   P=   0.22698D+01                MU=0.90   P=   0.12819D+01
            MU=1.00   P=   0.25503D+01                MU=1.00   P=   0.14322D+01
X=0.30                                    X=0.70
            MU=0.0    P=   0.0                        MU=0.0    P=   0.0
            MU=0.10   P=   0.16974D+00                MU=0.10   P=   0.13581D+00
            MU=0.20   P=   0.34620D+00                MU=0.20   P=   0.27200D+00
            MU=0.30   P=   0.53313D+00                MU=0.30   P=   0.40892D+00
            MU=0.40   P=   0.73074D+00                MU=0.40   P=   0.54687D+00
            MU=0.50   P=   0.93706D+00                MU=0.50   P=   0.68606D+00
            MU=0.60   P=   0.11496D+01                MU=0.60   P=   0.82665D+00
            MU=0.70   P=   0.13661D+01                MU=0.70   P=   0.96874D+00
            MU=0.80   P=   0.15850D+01                MU=0.80   P=   0.11123D+01
            MU=0.90   P=   0.18053D+01                MU=0.90   P=   0.12574D+01
            MU=1.00   P=   0.20263D+01                MU=1.00   P=   0.14039D+01
X=0.40                                    X=0.80
            MU=0.0    P=   0.0                        MU=0.0    P=   0.0
            MU=0.10   P=   0.15459D+00                MU=0.10   P=   0.13780D+00
            MU=0.20   P=   0.31146D+00                MU=0.20   P=   0.27593D+00
            MU=0.30   P=   0.47230D+00                MU=0.30   P=   0.41468D+00
            MU=0.40   P=   0.63786D+00                MU=0.40   P=   0.55434D+00
            MU=0.50   P=   0.80807D+00                MU=0.50   P=   0.69512D+00
            MU=0.60   P=   0.98237D+00                MU=0.60   P=   0.83719D+00
            MU=0.70   P=   0.11600D+01                MU=0.70   P=   0.98067D+00
            MU=0.80   P=   0.13401D+01                MU=0.80   P=   0.11256D+01
            MU=0.90   P=   0.15221D+01                MU=0.90   P=   0.12720D+01
            MU=1.00   P=   0.17055D+01                MU=1.00   P=   0.14199D+01
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X=0.90

          MU=0.0       P=    0.0
          MU=0.10      P=    0.14270D+00
          MU=0.20      P=    0.28574D+00
          MU=0.30      P=    0.42942D+00
          MU=0.40      P=    0.57403D+00
          MU=0.50      P=    0.71980D+00
          MU=0.60      P=    0.86693D+00
          MU=0.70      P=    0.10155D+01
          MU=0.80      P=    0.11657D+01
          MU=0.90      P=    0.13174D+01
          MU=1.00      P=    0.14705D+01
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X=1.00

          MU=0.0       P=    0.0
          MU=0.10      P=    0.15040D+00
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          MU=0.30      P=    0.45270D+00
          MU=0.40      P=    0.60529D+00
          MU=0.50      P=    0.75920D+00
          MU=0.60      P=    0.91465D+00
          MU=0.70      P=    0.10717D+01
          MU=0.80      P=    0.12305D+01
          MU=0.90      P=    0.13910D+01
          MU=1.00      P=    0.15529D+01
```

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BIOGRAPHICAL SKETCH

Lotfi Ammar was born January 1, 1954, in Sousse, Tunisia. He graduated from Sadiki High School in Tunis, June 1973. He attended the National Engineering School of Tunis from 1973 to 1979, receiving his degree in mechanical engineering in June 1979. He has been working on the degree of Doctor of Philosophy since 1980, while being a teaching assistant at the University of Florida.

He was married to Yasmina Kobbi in August 1981. One son, Sami, was born in January 1984.

Lotfi Ammar is a member of American Society of Mechanical Engineers and the American Society of Heating, Referigerating and Air Conditioning Engineers.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

C. K. Hsieh
Chung K. Hsieh, Chairman
Professor of Mechanical
Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

R.K. Irey
R.K. Irey
Professor of Mechanical
Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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Assistant Professor of
Mechanical Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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C.C. Hsu
Professor of Engineering
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This dissertation was submitted to the Graduate Faculty of the College of Engineering and to the Graduate School, and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

April 1984

Herbert A. Bevis

Dean, College of Engineering

Dean for Graduate Studies
and Research